# Topology

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# Contents

Chapter 1

# Set and Map

## 1.1 Set

Sets and elements are the most basic concepts of mathematics. Given any *element* $x$ and any *set* $X$, either $x$ belongs to $X$ (denoted $x \in X$), or $x$ does not belong to $X$ (denoted $x \notin X$). Elements are also figuratively called *points*.

**Example 1.1.1.** A set can be presented by listing all its elements.

1. $X_1 = \{1, 2, 3, \dots, n\}$ is the set of all integers between 1 and $n$.
2. $X_2 = \{2, -5\}$ is the set of all numbers satisfying the equation $x^2 + 3x - 10 = 0$.
3. $X_3 = \{a, b, c, \dots, x, y, z\}$ is the set of all latin alphabets.
4. $X_4 = \{\text{red, green, blue}\}$ is the set of basic colors that combine to form all the colors human can see.
5. $X_5 = \{\text{red, yellow}\}$ is the set of colors on the Chinese national flag.
6. The set $X_6$ of all registered students in the topology course is the list of names provided to me by the registration office.

**Example 1.1.2.** A set can also be presented by describing the properties satisfied by the elements.

1. Natural numbers $\mathbb{N} = \{n \colon n \text{ is obtained by repeatedly adding 1 to itself}\}$.
2. Prime numbers $\mathbb{P} = \{p \colon p \in \mathbb{N}, \text{ and the only integers dividing } p \text{ are } \pm 1 \text{ and } \pm p\}$.
3. Rational numbers $\mathbb{Q} = \{r \colon r \text{ is a quotient of two integers}\}$.
4. Open interval $(a, b) = \{x \colon x \in \mathbb{R}, a < x < b\}$.
5. Closed interval $[a, b] = \{x \colon x \in \mathbb{R}, a \le x \le b\}$.
6. Real polynomials $\mathbb{R}[t] = \{a_0 + a_1 t + a_1 t^2 + \cdots + a_n t^n \colon a_i \in \mathbb{R}\}$.
7. Continuous functions $C[0, 1] = \{f \colon \lim_{t \to a} f(t) = f(a) \text{ for any } 0 \le a \le 1\}$ on $[0, 1]$.
8. Unit sphere $S^2 = \{(x_1, x_2, x_3) \colon x_i \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$.
9. $X_1 = \{x \colon x \in \mathbb{N}, x \le n\}$, the first set in Example 1.1.1.
10. $X_2 = \{x \colon x^2 + 3x - 10 = 0\}$, the second set in Example 1.1.1.

Exercise 1.1.1. Present the following sets: the set $\mathbb{Z}$ of integers, the unit sphere $S^n$ in $\mathbb{R}^{n+1}$, the set $GL(n)$ of invertible $n \times n$ matrices, the set of latin alphabets in your name.

Exercise 1.1.2. Provide suitable names for the following subsets of $\mathbb{R}^2$.

1. $\{(x, y) \colon x = 0\}$.
2. $\{(x, y) \colon x = y\}$.
3. $\{(x, y) \colon x^2 + y^2 = 4\}$.
4. $\{(x, y) \colon x^2 + y^2 > 4\}$.
5. $\{(x, y) \colon y < 0\}$.
6. $\{(x, y) \colon x^2 + 4y^2 = 4\}$.
7. $\{(x, y) \colon |x| + |y| < 1\}$.
8. $\{(x, y) \colon |x| < 1, |y| < 1\}$.

A set $X$ is a *subset* of $Y$ if $x \in X$ implies $x \in Y$. In this case, we denote $X \subset Y$ and say "$X$ is *contained in* $Y$", or denote $Y \supset X$ and say "$Y$ *contains* $X$". The subset relation has the following properties:

- *reflexivity*: $X \subset X$.
- *antisymmetry*: $X \subset Y, Y \subset X \iff X = Y$.
- *transitivity*: $X \subset Y, Y \subset Z \implies X \subset Z$.

When the subsets are presented by properties satisfied by elements, the subset relation is the logical relation between the properties. For example, the subset relation $\{n\colon n^2 \text{ is even}\} \subset \{n\colon n \text{ is even}\}$ simply means "if $n^2$ is even, then $n$ is even".

The special notation $\emptyset$ is reserved for the *empty set*, the set with no element. The empty set is a subset of any set.

The *power set* $\mathcal{P}(X)$ of a set $X$, also denoted as $2^X$, is the collection of all subsets of $X$. For example, the power set of the set $\{1,2,3\}$ is

$$\mathcal{P}\{1,2,3\} = \big\{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\big\}.$$

The power set demonstrates that sets themselves can become elements of some other set, which we usually call a *collection* of sets. For example, the set

$$\big\{\emptyset,\ \{1,2\},\ \{1,3\},\ \{2,3\}\big\}$$

is the collection of subsets of $\{1,2,3\}$ with even number of elements.

Collections of sets appear quite often in everyday life and will play a very important role in the development of point set topology. For example, the set Parity = {Even, Odd} is actually the collection of two sets

$$\text{Even} = \{\dots,-4,-2,0,2,4,\dots\},\quad \text{Odd} = \{\dots,-3,-1,1,3,5,\dots\},$$

and the set Sign $= \{+,-,0\}$ is the collection of three sets

$$+ = \{x\colon x > 0\},\quad - = \{x\colon x < 0\},\quad 0 = \{0\}.$$

Exercise 1.1.3. How many elements are in the power set of $\{1,2,\dots,n\}$? (The answer suggests the reason for the notation $2^X$.) How many of these contain even number of elements?

Exercise 1.1.4. List all elements in $\mathcal{P}(\mathcal{P}\{1,2\})$, the power set of the power set of $\{1,2\}$.

Exercise 1.1.5. Show that the set of numbers $x$ satisfying $x^2 = 6x - 8$ is the same as the set of even integers between 1 and 5.

Exercise 1.1.6. For any real number $\epsilon > 0$, find a real number $\delta > 0$, such that

$$\{x\colon |x-1| < \delta\} \subset \{x\colon |x^2 - 1| < \epsilon\}.$$

Exercise 1.1.7. Find a number $n \in \mathbb{N}$, such that

$$\{m\colon m > n\} \subset \left\{m\colon \left|\frac{m}{m^2+1}\right| < 0.0001\right\}.$$

Exercise 1.1.8. For $r > 0$, let

$$D_r = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq r^2\},$$
$$S_r = \{(x, y) \in \mathbb{R}^2 \colon |x| \leq r, |y| \leq r\}.$$

What is the necessary and sufficient condition for $D_r \subset S_{r'}$? What is the necessary and sufficient condition for $S_{r'} \subset D_r$?

Exercise 1.1.9. A set $X$ is a *proper subset* of $Y$ if $X \subset Y$ and $X \neq Y$. Prove that if $X \subset Y \subset Z$, then the following are equivalent.

1. $X$ is a proper subset of $Z$.
2. Either $X$ is a proper subset of $Y$, or $Y$ is a proper subset of $Z$.

New sets can be constructed from the existing ones by the following basic operations:

- *union*: $X \cup Y = \{x \colon x \in X \text{ or } x \in Y\}$.
- *intersection*: $X \cap Y = \{x \colon x \in X \text{ and } x \in Y\}$.
- *difference*: $X - Y = \{x \colon x \in X \text{ and } x \notin Y\}$.
- *product*: $X \times Y = \{(x, y) \colon x \in X, y \in Y\}$.

Two sets $X$ and $Y$ are *disjoint* if $X \cap Y = \emptyset$. In other words, $X$ and $Y$ share no common element. The union of disjoint sets is sometimes denoted as $X \sqcup Y$, called *disjoint union*. If $Y \subset X$, then $X - Y$ is also called the *complement* of $Y$ in $X$. Moreover, $X^n$ denotes the product of $n$ copies of $X$.

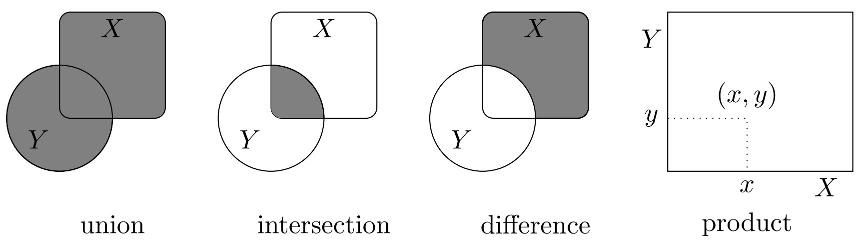



**Figure 1.1.1.** *set operations*

Some set operations can be extended to a collection $\{X_i \colon i \in I\}$ of sets:

- *union*: $\cup_i X_i = \{x \colon x \in X_i \text{ for some } i \in I\}$.
- *intersection*: $\cap_i X_i = \{x \colon x \in X_i \text{ for all } i \in I\}$.
- *product*: $\times_i X_i = \{(x_i)_{i \in I} \colon x_i \in X_i \text{ for each } i \in I\}$.

We also have the disjoint union $\sqcup_i X_i$ when the collection is pairwise disjoint: $X_i \cap X_j = \emptyset$ in case $i \neq j$.

The union and the intersection have the following properties:

- $X \cap Y \subset X \subset X \cup Y$, $X \cup \emptyset = X$, $X \cap \emptyset = \emptyset$.
- *commutativity*: $Y \cup X = X \cup Y$, $Y \cap X = X \cap Y$.
- *associativity*: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.
- *distibutivity*: $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$.

The properties are not hard to prove. For example, the proof of the associativity of the union involves the verification of $(X \cup Y) \cup Z \subset X \cup (Y \cup Z)$ and $(X \cup Y) \cup Z \supset X \cup (Y \cup Z)$. The first inclusion is verified below.

$$\begin{aligned}
x \in (X \cup Y) \cup Z &\implies x \in X \cup Y \text{ or } x \in Z \\
&\implies x \in X \text{ or } x \in Y \text{ or } x \in Z \\
&\implies x \in X \text{ or } x \in Y \cup Z \\
&\implies x \in X \cup (Y \cup Z).
\end{aligned}$$

The second inclusion can be verified similarly.

The properties for the union and the intersection can be extended to any (especially infinite) union and intersection.

The difference has the following properties:

- $X - Y = \emptyset \iff X \subset Y$.
- $X - Y = X \iff X \cap Y = \emptyset$.
- $Y \subset X \implies X - (X - Y) = Y$.
- *de Morgan's Law*[1]: $X - (Y \cup Z) = (X - Y) \cap (X - Z)$, $X - (Y \cap Z) = (X - Y) \cup (X - Z)$.

Note that de Morgan's law basically says that the complement operation exchanges the union and the intersection. The property can be extended to any union and intersection.

When several operations are mixed, we have a rule similar to the arithmetic operations. Usually the product $\times$ is taken first, then the union $\cup$ or the intersection $\cap$ is taken, and finally the difference $-$ is taken. For example, $x \in (X - Y \times Z) \cap W - U$ means $x \in X$, $x \notin Y \times Z$, $x \in W$, and $x \notin U$.

Exercise 1.1.10. What are the following sets?

1. $\{n : n \text{ is even}\} \cap \{n : n \text{ is divisible by } 5\}$.
2. $\{n : n \text{ is a positive integer}\} \cup \{n : n \text{ is a negative integer}\} \cup \{0\}$.
3. $\{n : n \text{ is even}\} \cup \{n : |n| < 10 \text{ and } n \text{ is an integer}\} - \{n : n \neq 2\} \cap \{n : n^2 \neq 6n - 8\}$.
4. $\{x : 1 < x \leq 10\} - \{x : 2x \text{ is an integer}\} \cap \{x : 6x^3 + 5x^2 - 33x + 18 = 0\}$.

[1]Augustus de Morgan, born June 27, 1806 in Madura, India, died March 18, 1871 in London, England. De Morgan introduced the term "mathematical induction" to put the proving method on a rigorous basis. His most important contribution is to the subject of formal logic.

5. $\{x\colon x > 0\} \times \{y\colon |y| < 1\} - \{(x, y)\colon x + y < 1\} - \{(x, y)\colon x > y\}$.
6. $\{(x, y)\colon x^2 + y^2 \le 1\} - \{x\colon x \ge 0\}^2$.

Exercise 1.1.11. Prove the properties of set operations.

1. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.
2. $(X \cap Y) \cup Z = (X \cup Y) \cap (X \cup Z)$.
3. $X - (Y \cap Z) = (X - Y) \cup (X - Z)$.
4. $X \cup Y - Z = (X - Z) \cup (Y - Z)$.
5. $(X - Y) \cup (Y - X) = X \cup Y - X \cap Y$.
6. $(X - Y) \cap (Y - X) = \emptyset$.
7. $X \times (Y \cap Z) = (X \cap Y) \times (X \cap Z)$.
8. $X \times Y - X \times Z = X \times (Y - Z)$.

Exercise 1.1.12. Find all the unions and intersections among $A = \{\emptyset\}$, $B = \{\emptyset, A\}$, $C = \{\emptyset, A, B\}$.

Exercise 1.1.13. Express the following using sets $X$, $Y$, $Z$ and operations $\cup$, $\cap$, $-$.

1. $A = \{x\colon x \in X \text{ and } (x \in Y \text{ or } x \in Z)\}$.
2. $B = \{x\colon (x \in X \text{ and } x \in Y) \text{ or } x \in Z\}$.
3. $C = \{x\colon x \in X,\ x \notin Y, \text{ and } x \in Z\}$.

Exercise 1.1.14. Let $A$ and $B$ be subsets of $X$. Prove that

$$A \subset B \iff X - A \supset X - B \iff A \cap (X - B) = \emptyset.$$

Exercise 1.1.15. Which statements are true?

1. $X \subset Z$ and $Y \subset Z \implies X \cup Y \subset Z$.
2. $X \subset Z$ and $Y \subset Z \implies X \cap Y \subset Z$.
3. $X \subset Z$ or $Y \subset Z \implies X \cup Y \subset Z$.
4. $Z \subset X$ and $Z \subset Y \implies Z \subset X \cap Y$.
5. $Z \subset X$ and $Z \subset Y \implies Z \subset X \cup Y$.
6. $Z \subset X \cap Y \implies Z \subset X$ and $Z \subset Y$.
7. $Z \subset X \cup Y \implies Z \subset X$ and $Z \subset Y$.
8. $X - (Y - Z) = (X - Y) \cup Z$.
9. $(X - Y) - Z = X - Y \cup Z$.
10. $X - (X - Z) = Z$.
11. $X \cap Y - Z = (X - Z) \cap (Y - Z)$.
12. $X \cap (Y - Z) = X \cap Y - X \cap Z$.
13. $X \cup (Y - Z) = X \cup Y - X \cup Z$.
14. $(X \cap Y) \cup (X - Y) = X$.
15. $X \subset U$ and $Y \subset V$
$\iff X \times Y \subset U \times V$.
16. $X \times (U \cup V) = X \times U \cup X \times V$.
17. $X \times (U - V) = X \times U - X \times V$.
18. $(X - Y) \times (U - V) = X \times U - Y \times V$.

Exercise 1.1.16. The union $X \cup Y$ of two sets $X$ and $Y$ is naturally divided into a union of three disjoint subsets $X - Y$, $Y - X$ and $X \cap Y$.

1. How many pieces can the union of three sets be naturally divided into?
2. Express the pieces in the first part in terms of the set operations. (One such piece is $X - (Y \cup Z)$, for example.)
3. How many pieces constitute the union of $n$ sets?

## 1.2 Map

A *map* from a set $X$ to a set $Y$ is a process $f$ that assigns, for each $x \in X$, a unique $y = f(x) \in Y$. The process should be *well-defined* in the following sense:

- *applicability*: The process can be applied to any input $x \in X$ to produce some output $f(x)$.

- *unambiguity*: For any input $x \in X$, the output $f(x)$ of the process is unique. In other words, the process $f$ is *single-valued*.

In case $Y$ is a set of numbers, the map $f$ is also called a *function*.

A map $f$ from $X$ to $Y$ is usually denoted as

$$f\colon X \to Y,\ x \mapsto f(x),$$

or

$$f(x) = y\colon X \to Y.$$

The sets $X$ and $Y$ are the *domain* and the *range* of the map. The point $y$ is the *image* (or the *value*) of $x$.

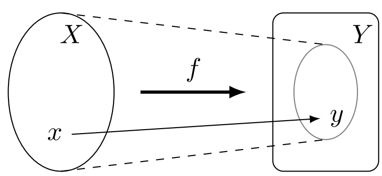



**Figure 1.2.1.** *domain, range, image*

**Example 1.2.1.** The map $f(x) = 2x^2 - 1\colon \mathbb{R} \to \mathbb{R}$ (equivalently, $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x^2 - 1$) means the following process: Multiply $x$ to itself to get $x^2$. Then multiply 2 to get $2x^2$. Finally subtract 1 to get $2x^2 - 1$. Since each step always works and gives unique outcome, the process is a map.

**Example 1.2.2.** The square root function $f(x) = \sqrt{x}\colon [0, \infty) \to \mathbb{R}$ is the following process: For any $x \geq 0$, find a *non-negative* number $y$, such that multiplying $y$ to itself yields $x$. Then $f(x) = y$. Again, since $y$ always exists and is unique, the process is a map.

If $[0, \infty)$ is changed to $\mathbb{R}$, then $y$ does not exist for negative $x$. So the applicability condition is violated, and the process $\sqrt{x}\colon \mathbb{R} \to \mathbb{R}$ is not a map.

If the process is modified by no longer requiring $y$ to be non-negative. Then the process can be applied to any $x \in [0, \infty)$, except there will be two outcomes (one positive, one negative) in general. So the unambiguity condition is violated, and the process is again not a map.

**Example 1.2.3.** The map $R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the process of rotating points on the plane around the origin by angle $\theta$ in the counter-clockwise direction.

**Example 1.2.4.** The map Age: $X_6 \to \mathbb{N}$ is the process of subtracting the birth year from the current year. The map Instructor: Courses $\to$ Professors takes the course to the professor who teaches the course. For example, the map "Instructor" takes "topology" to "me".

**Example 1.2.5.** For any set $X$, the *identity map* is

$$id_X(x) = x\colon X \to X,$$

and the *diagonal map* is

$$\Delta_X(x) = (x, x)\colon X \to X^2.$$

For any sets $X$, $Y$ and fixed $b \in Y$, the map

$$c(x) = b\colon X \to Y$$

is a *constant map.* Moreover, there are two *projection maps*

$$\pi_X(x, y) = x\colon X \times Y \to X, \quad \pi_Y(x, y) = y\colon X \times Y \to Y,$$

from the product of two sets.

Exercise 1.2.1. The following attempts to define a "square root" map. Which are actually maps?

1. For $x \in \mathbb{R}$, find $y \in \mathbb{R}$, such that $y^2 = x$. Then $f(x) = y$.
2. For $x \in [0, \infty)$, find $y \in \mathbb{R}$, such that $y^2 = x$. Then $f(x) = y$.
3. For $x \in [0, \infty)$, find $y \in [0, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
4. For $x \in [1, \infty)$, find $y \in [1, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
5. For $x \in [1, \infty)$, find $y \in (-\infty, -1]$, such that $y^2 = x$. Then $f(x) = y$.
6. For $x \in [0, 1)$, find $y \in [0, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
7. For $x \in [1, \infty)$, find $y \in (-\infty, -4] \cup [1, 2)$, such that $y^2 = x$. Then $f(x) = y$.

Exercise 1.2.2. Describe the processes that define the maps.

1. $2^n\colon \mathbb{Z} \to \mathbb{R}$.
2. $\sin\colon \mathbb{R} \to \mathbb{R}$.
3. Angle: Ordered pairs of straight lines in $\mathbb{R}^2 \to [0, 2\pi)$.
4. $\text{Area}_r$: Rectangles $\to [0, \infty)$.
5. $\text{Area}_t$: Triangles $\to [0, \infty)$.
6. Absolute value: $\mathbb{R} \to \mathbb{R}$.

Exercise 1.2.3. Suppose we want to combine two maps $f\colon X \to Z$ and $g\colon Y \to Z$ to get a new map $h\colon X \cup Y \to Z$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in X, \\ g(x), & \text{if } x \in Y. \end{cases}$$

What is the condition for $h$ to be a map?

The *image* of a subset $A \subset X$ under a map $f\colon X \to Y$ is

$$f(A) = \{f(a)\colon a \in A\} = \{y \in Y\colon y = f(a) \text{ for some } a \in A\}.$$

In the other direction, the *preimage* of a subset $B \subset Y$ is

$$f^{-1}(B) = \{x\colon f(x) \in B\}.$$

The image and the preimage have the following properties:

- $A \subset A' \implies f(A) \subset f(A')$.
- $B \subset B' \implies f^{-1}(B) \subset f^{-1}(B')$.
- $f(A \cup A') = f(A) \cup f(A')$.
- $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$.
- $f(A \cap A') \subset f(A) \cap f(A')$.
- $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$.
- $X - f^{-1}(B) = f^{-1}(Y - B)$.
- $A \subset f^{-1}(B) \iff f(A) \subset B$.

The properties can be extended to any union and intersection.

**Example 1.2.6.** Both the domain and the range of the map $f(x) = 2x^2 - 1\colon \mathbb{R} \to \mathbb{R}$ are $\mathbb{R}$. The image of the whole domain is $f(\mathbb{R}) = [-1, \infty)$. The image of $[0, \infty)$ is also $[-1, \infty)$, and the image of $[1, \infty)$ is $[1, \infty)$. The preimage of $[0, \infty)$ and 1 are respectively $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and $\{1, -1\}$.

Both the domain and the range of the rotation map $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ are $\mathbb{R}^2$. The image and the preimage of any circle centered at the origin are the circle itself. If the circle is not centered at the origin, then the image and the preimage are still circles, but at different locations.

The preimage $\text{Age}^{-1}(20)$ is all the 20 year old students in the class. The preimage $\text{Instructor}^{-1}(\text{me})$ is all the courses I teach.

Exercise 1.2.4. Find the image and the preimage of a straight line under the rotation map $R_\theta$? When is the image or the preimage the same as the original line?

Exercise 1.2.5. Show that in the property $f(A \cap A') \subset f(A) \cap f(A')$, the two sides are not necessarily the same.

Exercise 1.2.6. Which statements are true? If not, whether at least some directions or inclusions are true?

1. $A \cap A' = \emptyset \iff f(A) \cap f(A') = \emptyset$.
2. $B \cap B' = \emptyset \iff f^{-1}(B) \cap f^{-1}(B') = \emptyset$.

3. $f(A - A') = f(A) - f(A')$.
4. $f^{-1}(B - B') = f^{-1}(B) - f^{-1}(B')$.

Exercise 1.2.7. Prove that $f(f^{-1}(B)) = B \cap f(X)$. In particular, $f(f^{-1}(B)) = B$ if and only if $B \subset f(X)$.

Exercise 1.2.8. Prove that $f^{-1}(f(A)) = A$ if and only if $A = f^{-1}(B)$ for some $B \subset Y$.

Exercise 1.2.9. For a subset $A \subset X$, define the *characteristic function*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} : X \to \mathbb{R}.$$

1. Prove $\chi_{A\cap B} = \chi_A \chi_B$ and $\chi_{X-A} + \chi_A = 1$.
2. Express $\chi_{A\cup B}$ in terms of $\chi_A$ and $\chi_B$.
3. Describe the image and the preimage under the map $\chi_A$.

Exercise 1.2.10. Let $f\colon X \to Y$ be a map. Show that the "image map"

$$F\colon \mathcal{P}(X) \to \mathcal{P}(Y),\ A \mapsto f(A)$$

is indeed a map. Is the similar "preimage map" also a map?

A map $f\colon X \to Y$ is *onto* (or *surjective*) if $f(X) = Y$. This means that any $y \in Y$ is $f(x)$ for some $x \in X$. The map is *one-to-one* (or *injective*) if

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This is equivalent to

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

In other words, different elements have different images. The map is a *one-to-one correspondence* (or *bijective*) if it is one-to-one and onto.

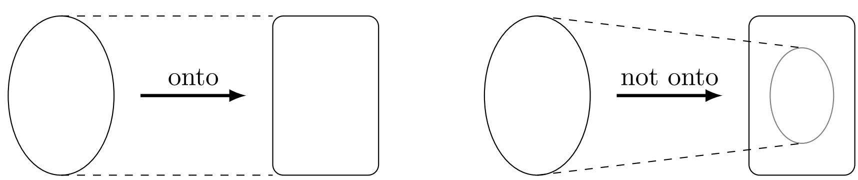



**Figure 1.2.2.** *onto and not onto*

For a map $f\colon X \to Y$, there are generally three possibilities at $y \in Y$:

1. *zero to one*: There is no $x \in X$ satisfying $f(x) = y$. This is the same as $y \notin f(X)$.
2. *one to one*: There is exactly one $x \in X$ satisfying $f(x) = y$.

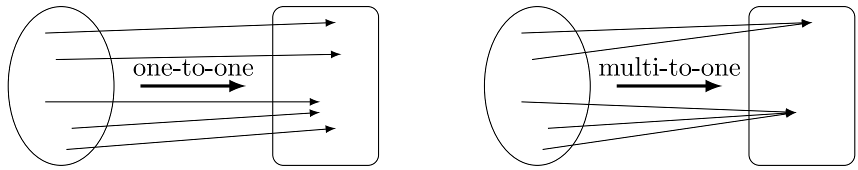



**Figure 1.2.3.** *one-to-one and multiple-to-one*

3. *many to one*: There are more than one $x \in X$ satisfying $f(x) = y$.

Clearly, onto means that the first possibility does not happen, and one-to-one means that the third possibility does not happen. Then one-to-one correspondence means that only the second possibility happens.

**Example 1.2.7.** The map $f(x) = 2x^2 - 1\colon \mathbb{R} \to \mathbb{R}$ is neither one-to-one nor onto. However, if the range is changed to $[-1, \infty)$, then the map $f(x) = 2x^2 - 1\colon \mathbb{R} \to [-1, \infty)$ is onto (although still not one-to-one).

The square root function $f(x) = \sqrt{x}\colon [0, \infty) \to \mathbb{R}$ is one-to-one but not onto. By changing the range to $[0, \infty)$, the function $f(x) = \sqrt{x}\colon [0, \infty) \to [0, \infty)$ becomes a one-to-one correspondence.

**Example 1.2.8.** The rotation map $R_\theta$ is a one-to-one correspondence.

The Age map is (almost certainly) not one-to-one and (definitely) not onto.

To say the map Instructor: Courses $\to$ Professors is onto means that every professor teaches some course. To say the map is one-to-one means that no professor teaches more than one courses.

Exercise 1.2.11. Which maps are onto? one-to-one?

1. $f(x) = (x, 2x)\colon \mathbb{R} \to \mathbb{R}^2$.
2. $f(\theta) = (\cos\theta, \sin\theta)\colon \mathbb{R} \to \mathbb{R}^2$.
3. $f(\theta) = (\cos\theta, \sin\theta)\colon [0, 2\pi) \to S^1 = \{(x, y)\colon x^2 + y^2 = 1\}$.
4. $f(x, y) = (2x - y - 1, 3x + 2y + 1)\colon \mathbb{R}^2 \to \mathbb{R}^2$.
5. $f(x, y) = (2x - y - 1, 3x + 2y + 1)\colon S^1 \to \mathbb{R}^2$.
6. $f(x, y) = (2x - y - 1, 3x + 2y + 1, x + y)\colon \mathbb{R}^2 \to \mathbb{R}^3$.
7. $f(x) = x^3 + x\colon \mathbb{R} \to \mathbb{R}$.
8. Integration: $C[0, 1] \to C[0, 1]$, $f(t) \mapsto \displaystyle\int_0^t f(x)dx$.
9. Sign: $\mathbb{R} \to \{+, 0, -\}$.
10. ID: Student $\to$ Number.
11. Poulation: City $\to$ Number.

Exercise 1.2.12. Prove the following properties about the preimages of single points.

1. $f\colon X \to Y$ is onto if and only if the preimage $f^{-1}(y)$ is not empty for any $y \in Y$.

2. $f\colon X \to Y$ is one-to-one if and only if the preimage $f^{-1}(y)$ contains at most one point for any $y \in Y$.

What is the criterion for a map to be a one-to-one correspondence in terms of the preimage?

Exercise 1.2.13. Is the "image map" in Exercise 1.2.10 onto or one-to-one? What about the "preimage map"? Will your conclusion be different if $f$ is assumed to be one-to-one or onto?

Exercise 1.2.14. After the domain or the range is reduced or enlarged, how are the onto and one-to-one properties changed?

Exercise 1.2.15. A map $f\colon X \to Y$ induces a map $\hat{f}\colon X \to f(X)$ by modifying the range from $Y$ to the image $f(X)$. Prove that

1. $\hat{f}$ is onto.
2. If $f$ is one-to-one, then $\hat{f}$ is a one-to-one correspondence.

Given two maps $f\colon X \to Y$ and $g\colon Y \to Z$ such that the range of $f$ and the domain of $g$ are the same, the *composition* $g \circ f$ (also denoted as $gf$) is

$$(g \circ f)(x) = g(f(x))\colon X \to Z.$$

The composition has the following properties:

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad id \circ f = f = f \circ id.$$

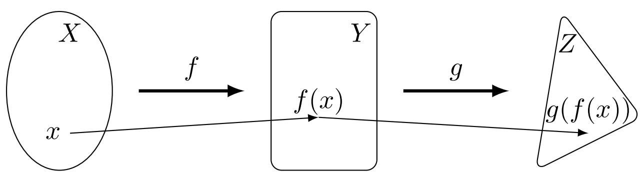



**Figure 1.2.4.** *composition $g \circ f$*

**Example 1.2.9.** For a subset $A \subset X$, we have the natural *inclusion map* $i(a) = a\colon A \to X$. For any map $f\colon X \to Y$, the composition $f \circ i\colon A \to Y$ is the *restriction* of $f$ on $A$ and is often denoted by $f|_A$.

**Example 1.2.10.** Since the composition of rotations by angles $\theta$ and $\theta'$ is the rotation by angle $\theta+\theta'$, we have $R_\theta \circ R_{\theta'} = R_{\theta+\theta'}$. Comparing the explicit formulae for $R_\theta(R_{\theta'}(x_1, x_2))$ and $R_{\theta+\theta'}(x_1, x_2)$, we get the addition formula for trigonometric functions:

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta',$$
$$\sin(\theta + \theta') = \cos\theta\sin\theta' + \sin\theta\cos\theta'.$$

Exercise 1.2.16. Prove that the preimage of the composition is computed by $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Exercise 1.2.17. Prove that if the composition $f \circ g$ is onto, then $f$ is onto. Prove that if the composition $f \circ g$ is one-to-one, then $g$ is one-to-one. What if the composition is a one-to-one correspondence?

A map $f\colon X \to Y$ is *invertible* if there is another map $g\colon Y \to X$ in the opposite direction, such that

$$g \circ f = id_X, \quad f \circ g = id_Y.$$

In other words,

$$g(f(x)) = x, \quad f(g(y)) = y.$$

The map $g$ is called the *inverse map* of $f$. The most important fact about invertible maps is the following.

**Proposition 1.2.1.** *A map is invertible if and only if it is a one-to-one correspondence.*

*Proof.* We prove one-to-one correspondence implies invertibility. The converse follows from Exercise 1.2.17 and the fact that the identity map is a one-to-one correspondence.

For a map $f\colon X \to Y$, define $g\colon Y \to X$ to be the following process: For any $y \in Y$, find $x \in X$ such that $f(x) = y$. Then take $g(y) = x$.

For the process to always go through means the existence of $x$ for any $y$. Such existence means exactly that $f$ is onto. For the process to produce a unique result means the uniqueness of $x$. Specifically, if we find $x_1, x_2$ such that $f(x_1) = y, f(x_2) = y$, then we must have $x_1 = x_2$. This is exactly the condition for $f$ to be one-to-one.

Thus $g$ is indeed a map. It is then easy to verify that it is indeed the inverse of $f$. □

Note that the notation $f^{-1}$ has been used to denote the preimage, which is defined for *all* maps. Thus writing $f^{-1}(B)$ does not necessarily imply that $f$ is invertible. Of course in the special case $f$ is indeed invertible, the preimage $f^{-1}(B)$ of a subset $B \subset Y$ under the map $f$ is the same as the image $f^{-1}(B)$ of the subset under the inverse map $f^{-1}$.

**Example 1.2.11.** The map $f(x) = 2x^2 - 1\colon \mathbb{R} \to \mathbb{R}$ is not invertible. However, if the domain and the range are changed to become $f(x) = 2x^2 - 1\colon [0, \infty) \to [-1, \infty)$, then the map is invertible.

The proof of Proposition 1.2.1 suggests the way to find the inverse map $g$: If $f(x) = 2x^2 - 1 = y$, then $g(y) = x$. Since the non-negative solution of the equation $2x^2 - 1 = y$ is $x = \sqrt{\dfrac{y+1}{2}}$, the inverse map is $g(y) = \sqrt{\dfrac{y+1}{2}}\colon [-1, \infty) \to [0, \infty)$.

**Example 1.2.12.** The process of rotating by angle $\theta$ is reversed by rotating by angle $-\theta$. Therefore $R_\theta$ is invertible, with $R_\theta^{-1} = R_{-\theta}$.

Exercise 1.2.18. Find the inverse maps.

1. $f(x,y) = (2x - y - 1, 3x + 2y + 1)\colon \mathbb{R}^2 \to \mathbb{R}^2$.
2. $f(x,y,z) = (x, x + y, x + y + z)\colon \mathbb{R}^3 \to \mathbb{R}^3$.
3. $f(x,y) = (2x + 1, 3x + x^2 - y)\colon \mathbb{R}^2 \to \mathbb{R}^2$.
4. $f(x) = x^4 + 4x^2 + 4\colon (-\infty, 0] \to [4, \infty)$.

Exercise 1.2.19. Prove that if $f$ and $g$ are invertible, then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 1.2.20. Suppose $f$ is invertible. Prove that $g \circ f$ is invertible if and only if $g$ is invertible.

Exercise 1.2.21. Note that the characteristic function in Exercise 1.2.9 is actually a map from $X$ to $\{0,1\}$. Since $\{0,1\}$ is a two-point set, the set of all maps from $X$ to $\{0,1\}$ is also denoted by $2^X$. Thus we have the "meta-characteristic map"

$$\chi\colon \mathcal{P}(X) \to 2^X, \ A \mapsto \chi_A.$$

Prove that the map

$$2^X \to \mathcal{P}(X), \ f \mapsto f^{-1}(1)$$

is the inverse of $\chi$. In particular, the number of subsets of $X$ is the same as the number of maps from $X$ to $\{0,1\}$. (Compare with Exercise 1.1.3)

## 1.3 Counting

We say there are 26 latin alphabets because we can "count" 26 of them. The counting process simply assigns a number to each alphabet

$$a \leftrightarrow 1, \ b \leftrightarrow 2, \ c \leftrightarrow 3, \ \dots, \ x \leftrightarrow 24, \ y \leftrightarrow 25, \ z \leftrightarrow 26.$$

The assignment is a one-to-one correspondence between the set $\{a, b, c, \dots, x, y, z\}$ and the set $\{1, 2, \dots, 26\}$. The example suggests the following basic definition of counting.

**Definition 1.3.1.** Two sets $X$ and $Y$ have the same *cardinality*, if there is an invertible map between $X$ and $Y$.

When two sets have the same cardinality, we also say that they have the "same number of elements". A set is *finite* if it is either empty or has the same cardinality as $\{1, 2, \dots, n\}$ for some natural number $n$. Such a set (if not empty) can be denoted as $\{x_1, x_2, \dots, x_n\}$. A set is *infinite* if it is not finite. A set is *countable* if it is either finite or has the same cardinality as the set $\mathbb{N}$ of natural numbers. A set of the same cardinality as $\mathbb{N}$ is *countably infinite* and can be denoted as $\{x_1, x_2, \dots\} = \{x_n \colon n \in \mathbb{N}\}$.

**Example 1.3.1.** The set $\mathbb{Z}$ of integers is countably infinite by the following one-to-one correspondence

$$f(a) = \begin{cases} 2a, & \text{if } a > 0 \\ 1 - 2a, & \text{if } a \le 0 \end{cases} : \mathbb{Z} \to \mathbb{N}.$$

Basically, this is the combination of a one-to-one correspondence between positive integers and even natural numbers and a one-to-one correspondence between nonpositive integers and odd natural numbers.

**Example 1.3.2.** Figure 1.3.1 shows a way of counting all the pairs of natural numbers. The exact formula for the counting method is

$$f(m, n) = \frac{1}{2}(m + n)(m + n - 1) - (m - 1) \colon \mathbb{N}^2 \to \mathbb{N}.$$

We conclude that $\mathbb{N}^2$ and $\mathbb{N}$ have the same number of elements. In general, $\mathbb{N}^k$ is countably infinite for any natural number $k$.

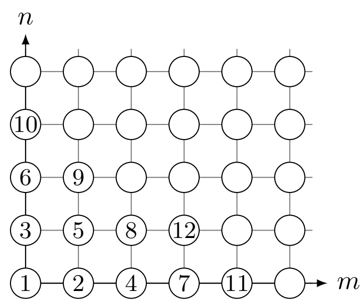



**Figure 1.3.1.** $\mathbb{N}^2$ *and* $\mathbb{N}$ *have the same number of elements.*

Exercise 1.3.1. Suppose $X$ and $Y$ have the same cardinality. Suppose $Y$ and $Z$ have the same cardinality. Prove that $X$ and $Z$ have the same cardinality.

Exercise 1.3.2. Use induction to prove that if there is a one-to-one correspondence between $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, then $m = n$. This implies that the number of elements in a finite set is a well-defined concept.

Exercise 1.3.3. Prove $\mathbb{N}$ is infinite by the following steps.

1. By inducting on $k$, prove the following statement: For any finitely many natural numbers $k_1, k_2, \dots, k_n$, there is a natural number $k$ such that $k > k_1$, $k > k_2$, ..., $k > k_n$.
2. Prove that no map $f \colon \{1, 2, \dots, n\} \to \mathbb{N}$ can be onto by taking $k_i = f(i)$ in the first part.

In general, it might be quite difficult to actually construct a suitable one-to-one correspondence. The following important result simplifies the task quite a lot.

**Theorem 1.3.2** (Schröder-Bernstein Theorem). *If there is a one-to-one map from $X$ to $Y$ and another one-to-one map from $Y$ to $X$, then $X$ and $Y$ have the same cardinality.*

*Proof.* Let $f\colon X \to Y$ and $g\colon Y \to X$ be one-to-one maps. Then

$$\phi = gf\colon X \to X, \quad \psi = fg\colon Y \to Y$$

are also one-to-one.

For a one-to-one map $f\colon X \to Y$, we have the following properties:

1. $f(A - B) = f(A) - f(B)$ for $B \subset A \subset X$.

2. For any $A \subset X$, the restriction $f|_A\colon A \to f(A)$ is a one-to-one correspondence.

Using the first property, we have disjoint union decompositions for $X$ and $Y$

$$\begin{aligned} X &= (X - \phi(X)) \sqcup (\phi(X) - \phi^2(X)) \sqcup (\phi^2(X) - \phi^3(X)) \sqcup \cdots \\ &= (X - \phi(X)) \sqcup \phi(X - \phi(X)) \sqcup \phi^2(X - \phi(X)) \sqcup \cdots \\ Y &= (Y - \psi(Y)) \sqcup \psi(Y - \psi(Y)) \sqcup \psi^2(Y - \psi(Y)) \sqcup \cdots . \end{aligned}$$

Moreover, we have the decompositions

$$\begin{aligned} X - \phi(X) &= (X - g(Y)) \sqcup (g(Y) - gf(X)) = A \sqcup g(B), \\ Y - \psi(Y) &= (Y - f(X)) \sqcup (f(X) - fg(Y)) = B \sqcup f(A), \end{aligned}$$

where

$$A = X - g(Y), \quad B = Y - f(X).$$

Combining the one-to-one correspondences $f|_A\colon A \to f(A)$ and $g|_B\colon B \to g(B)$, we get a one-to-one correspondence $h\colon X - \phi(X) \to Y - \psi(Y)$. Then the composition of one-to-one correspondences

$$\phi^n(X - \phi(X)) \xleftarrow{\phi^n} X - \phi(X) \xrightarrow{h} Y - \psi(Y) \xrightarrow{\phi^n} \psi^n(Y - \psi(Y))$$

is also a one-to-one correspondence between $\phi^n(X - \phi(X))$ and $\psi^n(Y - \psi(Y))$. Combining all these one-to-one correspondences for $n = 0, 1, 2, \ldots$, we get a one-to-one correspondence between $X$ and $Y$. $\square$

Theorem 1.3.2 allows us to define the following concept: The cardinality of $X$ is no bigger than (or $X$ has no more elements than) the cardinality of $Y$ if there is a one-to-one map $X \to Y$. In this case, we also say that the cardinality of $Y$ is no less than (or $Y$ has no fewer elements than) the cardinality of $X$. The theorem basically says that if $X$ has no more and no less elements than $Y$, then $X$ and $Y$ have the same number of elements. This is consistent with our usual understanding of the order relation, in which $x \leq y$ and $y \leq x$ implies $x = y$.

It is a consequence of the *axiom of choice*[2] that if there is an onto map $X \to Y$, then there is a one-to-one map $Y \to X$. Therefore $X$ contains no fewer elements than $Y$.

We remark that if there is a one-to-one map $X \to Y$, then there is an onto map $Y \to X$. The fact does not require the axiom of choice.

**Example 1.3.3.** We show that the rational numbers $\mathbb{Q}$ is countably infinite.

First, the natural inclusion $\mathbb{N} \to \mathbb{Q}$ implies that $\mathbb{Q}$ has no fewer elements than $\mathbb{N}$.

Second, any nonzero rational number $r$ can be uniquely written as $r = \dfrac{a}{b}$ with $b > 0$ and $a$, $b$ coprime, and we also write $0 = \dfrac{0}{1}$. Then

$$f(r) = (a, b) \colon \mathbb{Q} \to \mathbb{Z}^2.$$

is a one-to-one map. Furthermore, by Examples 1.3.1, 1.3.2 and Exercise 1.3.1, there is a one-to-one correspondence $g \colon \mathbb{Z}^2 \to \mathbb{N}$. Then the composition $gf \colon \mathbb{Q} \to \mathbb{N}$ is a one-to-one map. Thus $\mathbb{Q}$ has no more elements than $\mathbb{N}$.

By Theorem 1.3.2, we conclude that $\mathbb{Q}$ and $\mathbb{N}$ have the same number of elements.

**Example 1.3.4.** We show that the power set $\mathcal{P}(\mathbb{N})$ of the natural numbers $\mathbb{N}$ has the same number of elements as the real numbers $\mathbb{R}$.

Any real number $x \in (0, 1)$ has binary expansion

$$x = 0.a_1a_2a_3\cdots = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots, \quad a_i = 0 \text{ or } 1 \text{ and some } a_i \neq 0.$$

Such expansion is ambiguous when $a_j = 0$ and $a_i = 1$ for $i > j$. In this case, we use

$$\frac{1}{2^{j+1}} + \frac{1}{2^{j+2}} + \frac{1}{2^{j+3}} + \cdots = \frac{1}{2^j},$$

and choose $a_j = 1$ and $a_i = 0$ for $i > j$ instead. Then the expansion is unique for each $x$, and we get a well-defined one-to-one map

$$f(x) = \{i \colon a_i = 1\} \subset \mathbb{N} \colon (0, 1) \to \mathcal{P}(\mathbb{N}).$$

Combined with the one-to-one correspondence $\dfrac{1}{2} + \dfrac{1}{\pi} \arctan x \colon \mathbb{R} \to (0, 1)$, we get a one-to-one map $\mathbb{R} \to \mathcal{P}(\mathbb{N})$.

On the other hand, for any $A \subset \mathbb{N}$, we may construct a real number by decimal expansion

$$g(A) = \sum_{i \in A} \frac{1}{10^i} \colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}.$$

The use of decimal expansion makes sure that the map is one-to-one. Then by Theorem 1.3.2, we conclude that $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same number of elements.

Exercise 1.3.4. Prove that the subset of a countable set is countable.

Exercise 1.3.5. Prove that the countable union and finite product of countable sets are countable.

[2]The axiom of choice is a statement in set theory that cannot be proved or disproved. Mathematicians usually accept the axiom of choice as truth and use the axiom to prove theorems.

Exercise 1.3.6. Prove that the collection of all finite subsets of a countable set is countable.

Exercise 1.3.7. Suppose $A \subset \mathbb{N}$ is an infinite subset. Inductively define a map $f\colon \mathbb{N} \to A$ by

$$f(1) = \min A, \quad f(n+1) = \min(A - \{f(1), f(2), \dots, f(n)\}).$$

1. Prove that $f$ is defined for all natural numbers.
2. Prove that $f$ is one-to-one.
3. Prove that $A$ and $\mathbb{N}$ have the same number of elements.

In fact, $f$ is a one-to-one correspondence. Can you prove $f$ is onto?

Exercise 1.3.8. Prove that $\mathbb{R}^2$ and $\mathbb{R}$ have the same number of elements by the following steps.

1. Construct a one-to-one map from $(0,1)$ to $(0,1)^2$.
2. Using the idea $(0.123, 0.789) \mapsto 0.172839$, construct a one-to-one map from $(0,1)^2$ to $(0,1)$.
3. Prove that $(0,1)$ and $(0,1)^2$ have the same number of elements.

In general, $\mathbb{R}^n$ and $\mathbb{R}$ have the same number of elements.

Exercise 1.3.9. Prove that the following sets have the same number of elements as $\mathbb{R}$.

1. unit circle $\{(x,y) \in \mathbb{R}^2 \colon, x^2 + y^2 = 1\}$.
2. open unit disk $\{(x,y) \in \mathbb{R}^2 \colon, x^2 + y^2 < 1\}$.
3. closed square $\{(x,y) \in \mathbb{R}^2 \colon, |x| \le 1, |y| \le 1\}$.
4. unit sphere $\{(x,y,z) \in \mathbb{R}^3 \colon, x^2 + y^2 + z^2 = 1\}$.

**Theorem 1.3.3.** *For any set $X$, the power set $\mathcal{P}(X)$ has strictly more elements than $X$.*

We note that the single subset map

$$X \to \mathcal{P}(X),\ x \mapsto \{x\}$$

is one-to-one. Therefore $\mathcal{P}(X)$ contains no fewer elements than $X$. The key to the theorem is that no map $X \to \mathcal{P}(X)$ can be onto.

The theorem implies that there is no biggest set. By Example 1.3.4, the theorem also implies that $\mathbb{R}$ is *uncountably infinite*.

*Proof.* Suppose $f\colon X \to \mathcal{P}(X)$ is an onto map. Consider the subset

$$A = \{x \in X \colon x \notin f(x)\} \in \mathcal{P}(X).$$

Since $f$ is onto, we have $A = f(y)$ for some $y \in X$. Then two cases may happen:

1. If $y \in A$, then $y \in f(y)$. By the definition of $A$, this means that $y \notin A$, contradicting to the assumption.
2. If $y \notin A$, then $y \notin f(y)$. By the definition of $A$, this means that $y \in A$, contradicting to the assumption again.

Therefore we always get a contradiction, and $f$ can never be onto. $\square$

## 1.4 Equivalence Relation and Quotient

An *equivalence relation* on a set $X$ is a collection of pairs, denoted $x \sim y$ for $x, y \in X$, such that the following properties are satisfied:

- *symmetry*: $x \sim y \implies y \sim x$.
- *reflexivity*: $x \sim x$ for any $x \in X$.
- *transitivity*: $x \sim y$ and $y \sim z \implies x \sim z$.

**Example 1.4.1.** For integers $x, y \in \mathbb{Z}$, define $x \sim y$ if $x - y$ is even. This is an equivalence relation: Symmetry means that the negative of an even number is even. Reflexivity means that 0 is even. Transitivity means that the sum of even numbers is even.

On the other hand, define a new relation $x \sim y$ when $x - y$ is odd. The relation is not an equivalence relation because it is not symmetric and not transitive.

**Example 1.4.2.** For $x, y \in \mathbb{R}$, define $x \sim y$ if $x - y$ is an integer. This is an equivalence relation: Symmetry means that the negative of an integer is an integer. Reflexivity means that 0 is an integer. Transitivity means that the sum of integers is an integer.

On the other hand, define a new relation by $x \sim y$ when $x \le y$. This is not symmetric and is therefore not an equivalence relation.

**Example 1.4.3.** A map $f : X \to Y$ defines a relation $x_1 \sim x_2$ on $X$ when $f(x_1) = f(x_2)$. It is easy to see that this is an equivalence relation.

For example, the length map $(x, y) \to \sqrt{x^2 + y^2} : \mathbb{R}^2 \to \mathbb{R}$ defines an equivalence relation in which two vectors in $\mathbb{R}^2$ are related if they have the same length. The sign map $\mathbb{R} \to \{+, 0, -\}$ defines an equivalence relation in which two real numbers are related if they have the same sign.

**Example 1.4.4.** Let $X$ be all the people in the world. Consider the following relations:

1. *descendant relation*: $x \sim y$ if $x$ is a descendant of $y$.
2. *blood relation*: $x \sim y$ if $x$ and $y$ have the same ancesters.
3. *sibling relation*: $x \sim y$ if $x$ and $y$ have the same parents.

The descendant relation is neither reflexive nor symmetric, although it is transitive. The blood relation is reflexive and symmetric but not transitive. The sibling relation has all three properties and is an equivalence relation.

Exercise 1.4.1. Which are equivalence relations?

1. $X = \mathbb{R}$, $x \sim y$ if $|x| < 1$ and $|y| < 1$.
2. $X = \mathbb{R}$, $x \sim y$ if $|x - y| < 1$.
3. $X = \mathbb{R}^2$, $(x_1, x_2) \sim (y_1, y_2)$ if $x_1^2 + y_1 = x_2^2 + y_2$.
4. $X = \mathbb{R}^2$, $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 \le x_2$ or $y_1 \le y_2$.
5. $X = \mathbb{R}^2$, $(x_1, x_2) \sim (y_1, y_2)$ if $(x_1, x_2)$ is obtained from $(y_1, y_2)$ by some rotation around the origin.
6. $X = \mathbb{R}^2$, $(x_1, x_2) \sim (y_1, y_2)$ if $(x_1, x_2)$ is a scalar multiple of $(y_1, y_2)$.

7. $X = \mathbb{C}$, $x \sim y$ if $x - y$ is a real number.
8. $X = \mathbb{Z}$, $x \sim y$ if $x - y$ is a multiple of 5.
9. $X = \mathbb{Z} - \{0\}$, $x \sim y$ if $x = ky$ for some $k \in X$.
10. $X = \mathbb{Q} - \{0\}$, $x \sim y$ if $x = ky$ for some $k \in X$.
11. $X = \mathcal{P}(\{1, 2, \dots, n\})$, $A \sim B$ if $A \cap B \neq \emptyset$.
12. $X = \mathcal{P}(\{1, 2, \dots, n\})$, $A \sim B$ if $A \cap B = \emptyset$.

Exercise 1.4.2. Let $A \subset X$ be a subset. Then a relation $\sim$ on $X$ can be restricted to $A$ by defining $a \sim_A b$ for $a, b \in A$ if $a \sim b$ when $a$ and $b$ are considered as elements of $X$. Show that if $\sim$ is an equivalence relation on $X$, then $\sim_A$ is an equivalence relation on $A$.

Exercise 1.4.3. Suppose a relation $\sim$ on $X$ is reflexive and transitive. Prove that if we force the symmetry by adding $x \sim y$ (new relation) whenever $y \sim x$ (existing relation), then we get an equivalence relation.

Exercise 1.4.4. Suppose a relation $x \sim y$ has been defined for some (ordered) pairs of elements $(x, y)$ in $X$. Now introduce a new relation $\approx$ by $x \approx x$, and $x \approx y$ if there are $x = x_1, x_2, x_3, \dots, x_{n-1}, x_n = y$, such that for each $i$, we have either $x_i \sim x_{i+1}$ or $x_{i+1} \sim x_i$. Prove that $\approx$ is an equivalence relation. The equivalence relation $\approx$ is said to be *induced* from the relation $\sim$.

Given an equivalence relation $\sim$ on a set $X$ and an element $x \in X$, the subset

$$[x] = \{y \colon y \sim x\}$$

of all elements related to $x$ is called the *equivalence class* of $x$. The element $x$ is also called a *representative* of the equivalence class.

**Proposition 1.4.1.** *Any two equivalence classes are either identical or disjoint. In fact, we have*

$$\begin{aligned} [x] = [y] &\iff x \sim y, \\ [x] \cap [y] = \emptyset &\iff x \not\sim y. \end{aligned}$$

*Moreover, the whole set $X$ is the union of all equivalence classes.*

*Proof.* Suppose $x \sim y$. Then

$$\begin{aligned} z \in [x] &\implies z \sim x && \text{(by definition of } [x]\text{)} \\ &\implies z \sim y && \text{(by } x \sim y \text{ and transitivity)} \\ &\implies z \in [y]. && \text{(by definition of } [y]\text{)} \end{aligned}$$

This proves $[x] \subset [y]$. On the other hand, by symmetry, $x \sim y$ implies $y \sim x$. Thus we also have $[y] \subset [x]$. This completes the proof that $x \sim y$ implies $[x] = [y]$.

The following proves that $[x] \cap [y] \neq \emptyset$ implies $x \sim y$.

$$\begin{aligned} z \in [x] \cap [y] &\implies z \sim x, z \sim y && \text{(by definition of } [x] \text{ and } [y]) \\ &\implies x \sim z, z \sim y && \text{(by symmetry)} \\ &\implies x \sim y. && \text{(by transitivity)} \end{aligned}$$

This is equivalent to $x \not\sim y$ implying $[x] \cap [y] = \emptyset$.

Finally, we have $x \in [x]$ by the reflexivity. Therefore any element of $X$ is in some equivalence class. This implies that $X$ is equal to the union of all equivalence classes. $\square$

An equivalence relation $\sim$ on a set $X$ induces a collection (a set of subsets)

$$X/\!\sim\, = \{[x] \colon x \in X\}$$

of all equivalence classes, called the *quotient set*. Moreover, we have a natural onto map

$$q(x) = [x] \colon X \to X/\!\sim,$$

called the *quotient map*. Proposition 1.4.1 basically says that $x \sim y$ if and only if $q(x) = q(y)$.

Conversely, if $q \colon X \to \bar{X}$ is an onto map, then by the idea of Example 1.4.3, we get an equivalence relation $\sim$ on $X$ defined by

$$x \sim y \iff q(x) = q(y).$$

Since $q$ is onto, this definition means that the map

$$X/\!\sim\, \to \bar{X},\ [x] \mapsto q(x)$$

is a well-defined one-to-one correspondence. Since the sets $X/\!\sim$ and $\bar{X}$ are identified, we may call any onto map $q \colon X \to \bar{X}$ a quotient map and the corresponding range $\bar{X}$ the quotient set.

The discussion shows that the concept of equivalence relations on $X$ and the concept of quotient maps from $X$ are equivalent concepts.

**Proposition 1.4.2.** *Let $\sim$ be an equivalence relation on a set $X$. Let $q \colon X \to \bar{X}$ be the corresponding quotient map. Then the equation $\bar{f} \circ q = f$ gives a one-to-one correspondence between the maps $\bar{f} \colon \bar{X} \to Y$ from the quotient space and the maps $f \colon X \to Y$ satisfying*

$$x_1 \sim x_2 \implies f(x_1) = f(x_2).$$

In particular, the proposition says that if $f \colon X \to Y$ has the same value for equivalent elements in $X$, then there is a unique map $\bar{f} \colon \bar{X} \to Y$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[rr, "q"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "\bar{f}"] \\ & Y & \end{tikzcd}$$

*Proof.* If $f\colon X \to Y$ is given by $f = \bar{f} \circ q$, then

$$x_1 \sim x_2 \iff q(x_1) = q(x_2) \implies f(x_1) = \bar{f}(q(x_1)) = \bar{f}(q(x_1)) = f(x_2).$$

Conversely, suppose $f\colon X \to Y$ has the property in the proposition. Then we wish $\bar{f}(q(x)) = f(x)$ uniquely defines a map $\bar{f}\colon \bar{X} \to Y$. Specifically, the requirement $\bar{f}(q(x)) = f(x)$ means that $\bar{f}$ is the following process: For any $\bar{x} \in \bar{X}$, write $\bar{x} = q(x)$. Then $\bar{f}(\bar{x}) = f(x)$.

It remains to show that the process is well-defined. The process works for any $\bar{x}$ because $q$ is onto, so that suitable $x$ can always be found for $\bar{x}$. On the other hand, suppose we find two suitable $x_1, x_2 \in X$. Then $q(x_1) = q(x_2)$, which by the definition of the equivalence relation means $x_1 \sim x_2$. The condition on $f$ then implies $f(x_1) = f(x_2)$. This shows that the value of $\bar{f}$ is not ambiguous. $\square$

**Example 1.4.5.** In Example 1.4.1, we define two integers $x$ and $y$ to be equivalent if $x - y$ is even. In other words, equivalence relation means the same parity. Therefore the relation has two equivalence classes

$$[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}, \quad [1] = \{\dots, -3, -1, 1, 3, 5, \dots\}.$$

The two classes can be given more descriptive names "Even" and "Odd". The quotient map is the parity map

$$\text{Parity}\colon \mathbb{Z} \to \{\text{Even}, \text{Odd}\}.$$

**Example 1.4.6.** In Example 1.4.2, we define two real numbers $x$ and $y$ to be equivalent if $x - y$ is an integer. The corresponding equivalence classes are

$$[x] = x + \mathbb{Z} = \{\dots, x-2, x-1, x, x+1, x+2, \dots\}, \quad 0 \le x < 1.$$

The choice of the condition $0 \le x < 1$ is due to the fact that for any real number $y$, there is a unique $x$, such that $0 \le x < 1$ and $x - y$ is an integer. In fact, we have $x = y - n$, where $n$ is the biggest integer $\le y$. In particular, the quotient set $\mathbb{R}/\!\sim$ can be identified with the interval $[0, 1)$, and the quotient map is $q(y) = y - n\colon \mathbb{R} \to [0, 1)$.

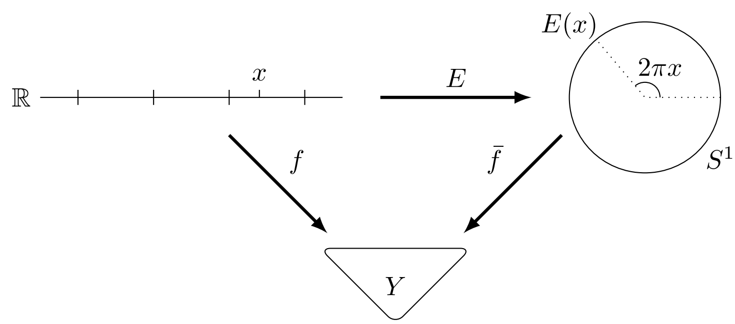



**Figure 1.4.1.** *quotient of* $\mathbb{R}$ *and periodic maps*

A better choice of the quotient set is the unit circle $S^1$ on the plane, which is also all the complex numbers of norm 1. The map $E(x) = e^{2\pi i x}\colon \mathbb{R} \to S^1$ is onto and satisfies

$$E(x) = E(y) \iff 2\pi x - 2\pi y \in 2\pi\mathbb{Z} \iff x - y \in \mathbb{Z}.$$

Therefore $E$ is also the quotient map corresponding to the equivalence relation.

Proposition 1.4.2 means that a map $f\colon \mathbb{R} \to Y$ satisfies

$$x - y \in \mathbb{Z} \implies f(x) = f(y)$$

if and only if $f(x) = \bar{f}(E(x)) = \bar{f}(e^{2\pi ix})$ for some map $\bar{f}\colon S^1 \to Y$. The condition on $f$ is the same as $f(x+1) = f(x)$. Therefore, we conclude that real periodic functions with period 1 are in one-to-one correspondence with functions on the unit circle.

**Example 1.4.7.** If $Y \neq \emptyset$, then the projection $\pi_X\colon X \times Y \to X$ is onto. The equivalence relation induced by the projection is

$$(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2.$$

Then Proposition 1.4.2 says that $f\colon X \times Y \to Z$ satisfies $f(x, y_1) = f(x, y_2)$ for any $x \in X$ and $y_1, y_2 \in Y$ if and only if $f(x, y) = \bar{f}(x)$. In other words, $f(x, y)$ does not depend on the second variable.

Exercise 1.4.5. On any set $X$, define $x \sim y$ if $x = y$. Show that this is an equivalence relation. What is the corresponding quotient?

Exercise 1.4.6. For the equivalence relations in Exercise 1.4.1, find the corresponding equivalence classes and quotients.

Exercise 1.4.7. Let $\sim_1$ and $\sim_2$ be two equivalence relations on $X$, such that

$$x \sim_1 y \implies x \sim_2 y.$$

How are the quotient sets $X/{\sim_1}$ and $X/{\sim_2}$ related?

Exercise 1.4.8. Let $\sim_X$ and $\sim_Y$ be equivalence relations on $X$ and $Y$. A map $f\colon X \to Y$ is said to *preserve the relation* if

$$x_1 \sim_X x_2 \implies f(x_1) \sim_Y f(x_2).$$

Prove that such a map induces a unique map $\bar{f}\colon X/{\sim_X} \to Y/{\sim_Y}$ such that $q_Y \circ f = \bar{f} \circ q_X$. In other words, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q_X"'] & Y \arrow[d, "q_Y"] \\
X/{\sim_X} \arrow[r, "\bar{f}"] & Y/{\sim_Y}
\end{tikzcd}$$

Exercise 1.4.9. Define two nonzero vectors $u$ and $v$ in a real vector space $V$ to be equivalent if $x = ry$ for some real number $r > 0$. Show that the unit sphere $SV$ of the vector space can be naturally identified with the quotient set of nonzero vectors under the relation. Then use this to identify homogeneous functions (with fixed degree) on $V - \{0\}$ with functions on the unit sphere.

Exercise 1.4.10. Let $V$ be a vector space. Let $H \subset V$ be a vector subspace. For $x, y \in V$, define $x \sim y$ if $x - y \in H$.

1. Prove that $\sim$ is an equivalence relation on $V$.
2. Construct additions and scalar multiplications on the quotient set $V/\sim$ so that it is also a vector space.
3. Prove that the quotient map $V \to V/\sim$ is a linear transformation.

The quotient vector space $V/\sim$ is usally denoted by $V/H$.

Exercise 1.4.11. Let $\mathcal{F}$ be the collection of all finite sets. For $A, B \in \mathcal{F}$, define $A \sim B$ if there is a one-to-one correspondence $f \colon A \to B$. Prove that this is an equivalence relation. Moreover, identify the quotient set as the set of non-negative integers and the quotient map as the number of elements in a set. The exercise leads to a general theory of counting.

Exercise 1.4.12. A *partition* of a set $X$ is a decomposition into a disjoint union of nonempty subsets

$$X = \sqcup_{i \in I} X_i, \quad X_i \neq \emptyset.$$

1. Define $x \sim y$ if $x$ and $y$ are in the same subset $X_i$. Prove that this is an equivalence relation.
2. Prove that $X_i$ are exactly the equivalence classes for the equivalence relation in the first part.
3. Prove that equivalence relations on $X$ are in one-to-one correspondence with partitions of $X$.

Exercise 1.4.13. Exercise 1.4.12 shows that the concept of equivalence relations on $X$ and the concept of partitions of $X$ are equivalent concepts. Explain how the concept of quotient maps from $X$ and the concept of partitions of $X$ are equivalent.

Exercise 1.4.14. Find the partitions for the equivalence relations in Exercise 1.4.1.

Exercise 1.4.15. For the equivalence relation on $X$ induced by a map $f \colon X \to Y$ in Example 1.4.3, show that the equivalence classes are $[x] = f^{-1}(f(x))$, and the corresponding partition is $X = \sqcup_{y \in f(X)} f^{-1}(y)$.

Exercise 1.4.16. Find the quotient maps and the equivalence relations corresponding to the partitions.

1. $\mathbb{Z} = \{3n \colon n \in \mathbb{Z}\} \sqcup \{3n+1 \colon n \in \mathbb{Z}\} \sqcup \{3n+2 \colon n \in \mathbb{Z}\}$.
2. $\mathbb{R}^2 = \sqcup_{r \geq 0}(\text{circle of radius } r)$.
3. $\mathbb{R} = (0, \infty) \sqcup \{0\} \sqcup (-\infty, 0)$.
4. $X \times Y = \sqcup_{x \in X} x \times Y$.

Chapter 2

# Metric Space

## 2.1 Metric

Recall that a function $f(x)\colon \mathbb{R} \to \mathbb{R}$ is continuous at $a$ if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Loosely speaking, this means

$$x \text{ is close to } a \implies f(x) \text{ is close to } f(a).$$

Since the closeness is measured by the distance $d(u, v) = |u - v|$ between numbers, the notion of continuity can be extended by considering other distances. For example, by introducing the Euclidean distance in $\mathbb{R}^n$,

$$d((x_1, \dots, x_n), (a_1, \dots, a_n)) = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$$

the continuity of a multivariable function $f(x_1, \dots, x_n)\colon \mathbb{R}^n \to \mathbb{R}$ may be defined. For another example, define the distance

$$d(f(t), g(t)) = \max_{0 \le t \le 1} |f(t) - g(t)|$$

between two continuous functions on $[0, 1]$. Then for the integration map $\mathcal{I}\colon f(t) \mapsto \int_0^1 f(t)dt$, by taking $\delta = \epsilon$, we have

$$d(f, g) < \delta \implies |\mathcal{I}(f) - \mathcal{I}(g)| \le \int_0^1 |f(t) - g(t)|dt < \delta = \epsilon.$$

This shows that $\mathcal{I}$ is a continuous map.

**Definition 2.1.1.** A *metric space* is a set $X$, together with a *metric* (also called *distance*)

$$d\colon X \times X \to \mathbb{R},$$

satisfying the following properties:

- *positivity*: $d(x, y) \ge 0$, and $d(x, y) = 0$ if and only if $x = y$.
- *symmetry*: $d(x, y) = d(y, x)$.
- *triangle inequality*: $d(x, y) + d(y, z) \ge d(x, z)$.

**Example 2.1.1.** On any set $X$, the function

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \ne y. \end{cases}$$

is a metric, called the *discrete metric*.

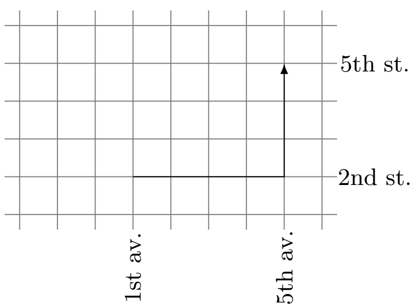
**Example 2.1.2.** The subway fare $d(x,y)$ from station $x$ to station $y$ is a metric on the set $X$ of all the subway stations in Hong Kong because it has the three properties. However, if $X$ is changed to all the bus stops in Hong Kong and $d(x,y)$ is the lowest bus fare (adding fares for several bus rides if necessary) from stop $x$ to stop $y$, then $d$ satisfies the positivity and the triangle inequality but fails the symmetry. Therefore the lowest bus fare in Hong Kong is not a metric. Note that the bus fares are calculated in different ways in different cities. In cities where the bus fare is independent of the length and the direction of the ride, the lowest bus fare is indeed a metric.

**Example 2.1.3.** The following are some commonly used metrics on $\mathbb{R}^n$:

$$\begin{aligned}
d_2((x_1,\dots,x_n),(y_1,\dots,y_n)) &= \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}, && \textit{Euclidean metric}\\
d_1((x_1,\dots,x_n),(y_1,\dots,y_n)) &= |x_1-y_1|+\cdots+|x_n-y_n|, && \textit{taxicab metric}\\
d_\infty((x_1,\dots,x_n),(y_1,\dots,y_n)) &= \max\{|x_1-y_1|,\dots,|x_n-y_n|\}. && L_\infty\textit{-metric}
\end{aligned}$$

More generally, for $p \ge 1$, we have

$$d_p((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt[p]{|x_1-y_1|^p+\cdots+|x_n-y_n|^p}. \qquad L_p\textit{-metric}$$




To go from 1st av. and 2nd st. to 5th av. and 5th st., one needs to drive 7 blocks.

**Figure 2.1.1.** *taxicab metric*

To prove the triangle inequality for $d_p$ in case $p \ge 1$, we use the convexity of $t^p$ for non-negative $t$ to get

$$(\lambda t+(1-\lambda)s)^p \le \lambda t^p+(1-\lambda)s^p, \quad \text{for } 0\le\lambda\le 1 \text{ and } t,s\ge 0.$$

Substituting

$$t=\frac{|x_i-y_i|}{d_p(x,y)}, \quad s=\frac{|y_i-z_i|}{d_p(y,z)}, \quad \lambda=\frac{d_p(x,y)}{d_p(x,y)+d_p(y,z)}$$

into the inequality gives us

$$\frac{|x_i-z_i|^p}{(d_p(x,y)+d_p(y,z))^p} \le \frac{(|x_i-y_i|+|y_i-z_i|)^p}{(d_p(x,y)+d_p(y,z))^p} \le \frac{d_p(x,y)\dfrac{|x_i-y_i|^p}{d_p(x,y)^p}+d_p(y,z)\dfrac{|y_i-z_i|^p}{d_p(y,z)^p}}{d_p(x,y)+d_p(y,z)}.$$

Taking sum over $i=1,\dots,n$, we get

$$\frac{d_p(x,z)^p}{(d_p(x,y)+d_p(y,z))^p} \le \frac{d_p(x,y)\dfrac{d_p(x,y)^p}{d_p(x,y)^p}+d_p(y,z)\dfrac{d_p(y,z)^p}{d_p(y,z)^p}}{d_p(x,y)+d_p(y,z)} = 1.$$

Then taking the $p$-th root gives us the triangle inequality.

**Example 2.1.4.** On the set $C[0,1]$ of all continuous functions on $[0,1]$, we have the following metrics similar to the Euclidean space:

$$\begin{aligned} d_1(f,g) &= \int_0^1 |f(t)-g(t)|dt, && L_1\textit{-metric} \\ d_\infty(f,g) &= \max_{0\le t\le 1} |f(t)-g(t)|, && L_\infty\textit{-metric} \\ d_p(f,g) &= \sqrt[p]{\int_0^1 |f(t)-g(t)|^p dt}. && L_p\textit{-metric} \end{aligned}$$

**Example 2.1.5.** Fix a prime number $p$. The *$p$-adic metric* on the set $\mathbb{Q}$ of rational numbers is defined as follows: For distinct rational numbers $x$ and $y$, the difference can be uniquely expressed as

$$x-y=\frac{m}{n}p^a,$$

where $m$, $n$, $a$ are integers, and neither $m$ nor $n$ is divisible by $p$. Define

$$d_{p\text{-adic}}(x,y)=\frac{1}{p\text{-factor of }(x-y)}=p^{-a}.$$

In addition, define $d_{p\text{-adic}}(x,x)=0$ for the case $x=y$. For example, for $x=\dfrac{7}{6}$ and $y=\dfrac{3}{4}$, we have

$$x-y=\frac{5}{12}=2^{-2}\cdot 3^{-1}\cdot 5.$$

Therefore

$$\begin{aligned} \text{2-factor of } (x-y) \text{ is } 2^{-2} &\implies d_{2\text{-adic}}(x,y)=2^2=4, \\ \text{3-factor of } (x-y) \text{ is } 3^{-1} &\implies d_{3\text{-adic}}(x,y)=3^1=3, \\ \text{5-factor of } (x-y) \text{ is } 5^{1} &\implies d_{5\text{-adic}}(x,y)=5^{-1}=0.2, \\ \text{7-factor of } (x-y) \text{ is } 7^{0} &\implies d_{7\text{-adic}}(x,y)=7^{-0}=1. \end{aligned}$$

We need to show that the $p$-adic metric has the three properties. The only nontrivial part is the triangle inequality. In fact, we will prove a stronger inequality

$$d_{p\text{-adic}}(x,z)\le \max\{d_{p\text{-adic}}(x,y), d_{p\text{-adic}}(y,z)\}.$$

It is left to you to show that this implies the triangle inequality.

The inequality is clearly true when at least two of $x,y,z$ are the same. Assume $x,y,z$ are distinct and $d_{p\text{-adic}}(x,y)=p^{-a}$, $d_{p\text{-adic}}(y,z)=p^{-b}$. Then

$$x-y=\frac{m}{n}p^a,\quad y-z=\frac{k}{l}p^b,$$

and none of $k,l,m,n$ is divisible by $p$. Without loss of generality, we may assume $a\le b$. Then

$$x-z=\frac{m}{n}p^a+\frac{k}{l}p^b=\frac{ml+nkp^{b-a}}{nl}p^a.$$

Since $a\le b$, the numerator of the fraction is an integer, which may contribute a $p$-factor $p^c$ only for some $c\ge 0$. Moreover, the denominator contributes no powers of $p$ because neither $n$ nor $l$ is divisible by $p$. Therefore the $p$-factor of $x-z$ is $p^c p^a = p^{c+a}$ for some $c \geq 0$, and

$$d_{p\text{-adic}}(x,z) = p^{-(c+a)} \leq p^{-a} = \max\{p^{-a}, p^{-b}\},$$

where the last equality comes from the assumption $a \leq b$.

Exercise 2.1.1. Suppose $d(1,2) = 2$ and $d(1,3) = 3$. What number can $d(2,3)$ be so in order for $X = \{1,2,3\}$ to be a metric space?

Exercise 2.1.2. Explain why, for $p < 1$, the formula for the $L_p$-metric in Example 2.1.3 no longer defines a metric on $\mathbb{R}^n$.

Exercise 2.1.3. Prove that the Euclidean metric and the taxicab metric satisfy $d_2(x,y) \leq d_1(x,y) \leq \sqrt{n}d_2(x,y)$. Find similar inequalities between other pairs of metrics on $\mathbb{R}^n$.

Exercise 2.1.4. Prove $d_\infty(x,y) = \lim_{p\to\infty} d_p(x,y)$. This explains the term "$L_\infty$-metric".

Exercise 2.1.5. Prove that the $L_p$-metric on $C[0,1]$ in Example 2.1.4 satisfies the triangle inequality for $p \geq 1$.

Exercise 2.1.6. Prove that any metric satisfies the inequalities

$$|d(x,y) - d(y,z)| \leq d(x,z), \quad |d(x,y) - d(z,w)| \leq d(x,z) + d(y,w).$$

Exercise 2.1.7. Prove that if $d(x,y)$ satisfies the positivity and the triangle inequality, then $\bar{d}(x,y) = d(x,y) + d(y,x)$ is a metric. This suggests that the symmetry condition is relatively less important.

Exercise 2.1.8. Prove that the three conditions for metric is equivalent to the following two:

- $d(x,y) = 0$ if and only if $x = y$.
- $d(y,x) + d(y,z) \geq d(x,z)$.

Exercise 2.1.9. Suppose a function $d$ on $X$ satisfies all three conditions for metric except "$d(x,y) = 0$ if and only if $x = y$". Prove that the condition $d(x,y) = 0$ defines an equivalence relation $x \sim y$ on $X$, and $d$ induces a well-defined metric on the quotient set.

Exercise 2.1.10. Which among $(x-y)^2$, $\sqrt{|x-y|}$, $|x^2 - y^2|$ are metrics on $\mathbb{R}$?

Exercise 2.1.11. Suppose $d$ is a metric. Which among $\min\{d,1\}$, $\dfrac{d}{1+d}$, $d^2$, $\sqrt{d}$ are also metrics?

Exercise 2.1.12. Suppose $d_1$ and $d_2$ are metrics. Which among $\max\{d_1,d_2\}$, $\min\{d_1,d_2\}$, $d_1 + d_2$, $|d_1 - d_2|$ are also metrics?

Exercise 2.1.13. Given metrics on $X$ and $Y$, construct at least two metrics on the product $X \times Y$.

Exercise 2.1.14. Study functions $f(t)$ with the property that $d(x, y)$ is a metric implies that $f(d(x, y))$ is also a metric.

1. Prove that if $f(0) = 0$, $f(t) > 0$ for $t > 0$, $f(s) + f(t) \geq f(s+t)$ and $f$ is increasing, then $f$ has the property.
2. Prove that if a concave function satisfies $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then it also satisfies the other two conditions in the first part. On the other hand, construct a non-concave function satisfying the conditions in the first part.
3. Prove that if $f(0) = 0$ and $1 \leq f(t) \leq 2$ for $t > 0$, then $f$ has the property.
4. Prove that if a function has the property and is continuous at 0, then the function is continuous everywhere.

Exercise 2.1.15. The power set $\mathcal{P}(\mathbb{N})$ is the collection of all subsets of the set $\mathbb{N}$ of natural numbers. For example, the following are considered as three points in $\mathcal{P}(\mathbb{N})$:

$$\begin{aligned} E &= \{2, 4, 6, 8, \dots\} = \text{all even numbers}, \\ P &= \{2, 3, 5, 7, \dots\} = \text{all prime numbers}, \\ S &= \{1, 4, 9, 16, \dots\} = \text{all square numbers}. \end{aligned}$$

For $A, B \in \mathcal{P}(\mathbb{N})$, define

$$d(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \dfrac{1}{\min((A - B) \cup (B - A))}, & \text{if } A \neq B. \end{cases}$$

For example, $(E - P) \cup (P - E) = \{3, 4, 5, \dots\}$ implies $d(E, P) = \dfrac{1}{3}$. Prove

1. $d(A, C) \leq \max\{d(A, B), d(B, C)\}$.
2. $d$ is a metric.
3. $d(\mathbb{N} - A, \mathbb{N} - B) = d(A, B)$.

The last property means that the complement map $A \mapsto \mathbb{N} - A$ is an *isometry*.

## 2.2 Ball

Let $(X, d)$ be a metric space. The (open) *ball* of radius $\epsilon > 0$ centered at $a \in X$ is

$$B_d(a, \epsilon) = \{x \colon d(x, a) < \epsilon\}.$$

By using balls, the closeness between two points can be rephrased in set-theoretical language

$$d(x, y) < \epsilon \iff x \in B_d(y, \epsilon) \iff y \in B_d(x, \epsilon).$$

A ball in the discrete metric is

$$B(a, \epsilon) = \begin{cases} \{a\}, & \text{if } \epsilon \leq 1, \\ X, & \text{if } \epsilon > 1. \end{cases}$$

Figure 2.2.1 shows the balls in various metrics on $\mathbb{R}^2$. In $C[0, 1]$, $g \in B_{L_\infty}(f, \epsilon)$ means

$$|f(t) - g(t)| < \epsilon \text{ for all } 0 \leq t \leq 1,$$

as illustrated in Figure 2.2.2. The ball $B_{2\text{-adic}}(0,1)$ in $\mathbb{Q}_{2\text{-adic}}$ consists of all rational numbers $\dfrac{m}{n}$ with $m$ even and $n$ odd.

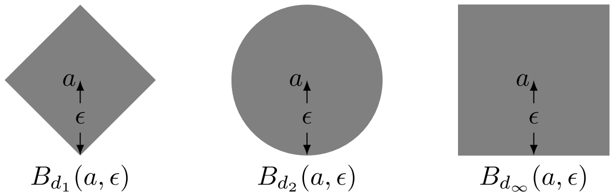



**Figure 2.2.1.** *balls in different metrics of* $\mathbb{R}^2$

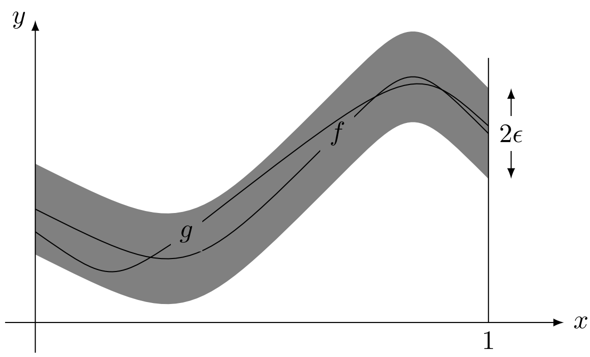



**Figure 2.2.2.** $g \in B_{L_\infty}(f,\epsilon)$

Exercise 2.2.1. For the Euclidean metric on $\mathbb{R}^2$, find the biggest $\epsilon$ such that the ball $B((1,1),\epsilon)$ is contained in both $B((0,0),2)$ and $B((2,1),2)$. Moreover, do the problem again with the $L_\infty$-metric in place of the Euclidean metric.

Exercise 2.2.2. For any $x \in \mathbb{R}^n$ and $\epsilon > 0$, find $\delta > 0$, such that $B_{L_2}(x,\delta) \subset B_{L_1}(x,\epsilon)$. Moreover, do the problem again for other pairs of metrics on $\mathbb{R}^n$.

Exercise 2.2.3. The Euclidean metric has the property that $B(x_1,\epsilon_1) = B(x_2,\epsilon_2)$ if and only if $x_1 = x_2$ and $\epsilon_1 = \epsilon_2$. Is this also true for any metric?

Exercise 2.2.4. Among the functions 0, $t$ and $t+\dfrac{1}{2}$, which are in the ball $B_{L_1}(1,1)$? Which are in $B_{L_\infty}(1,1)$? Moreover, can you find a function in $B_{L_\infty}(1,1) - B_{L_1}(1,1)$?

Exercise 2.2.5. Describe the numbers in $B_{2\text{-adic}}(0,2)$ and $B_{2\text{-adic}}(0,2) - B_{2\text{-adic}}(0,1)$. Moreover, prove $B_{2\text{-adic}}(0,3) = B_{2\text{-adic}}(0,4)$.

Exercise 2.2.6. What is the ball $B(E, 0.1)$ in the metric space in Exercise 2.1.15? Moreover, prove that if $A \subset \mathbb{N}$ is finite, then $C \in B\left(A, \dfrac{1}{\max A}\right)$ implies $A \subset C$.

Exercise 2.2.7. Let $A$ be a subset in a metric space $X$. Prove the following are equivalent.

1. $A$ is contained in a ball.
2. For any $a \in X$, $A$ is contained in a ball centered at $a$.
3. There is a number $D$, such that $d(x, y) \leq D$ for any $x, y \in A$.

The subset $A$ is called *bounded*. The smallest $D$ in the third condition is the *diameter* of the bounded subset.

Exercise 2.2.8. A metric satisfying the inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ is called an *ultrametric*. The $p$-adic metric is an ultrametric. Prove that if $d$ is an ultrametric and $d(x, y) < \epsilon$, then $B_d(x, \epsilon) = B_d(y, \epsilon)$.

Exercise 2.2.9. Prove that metric spaces are *Hausdorff*[3]: For $x \neq y$, there is $\epsilon > 0$, such that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$.

## 2.3 Open Subset

**Definition 2.3.1.** A subset $U$ of a metric space $X$ is *open* if for any $a \in U$, we have $B(a, \epsilon) \subset U$ for some $\epsilon > 0$.

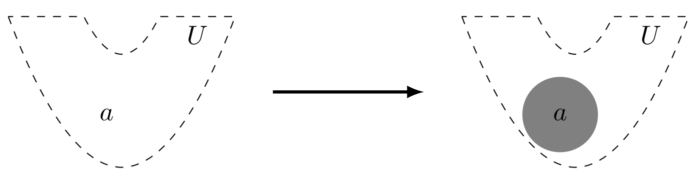



**Figure 2.3.1.** *definition of open subset*

Note that open subsets are defined by means of (the system of) balls. In terms of the metric, the openness of $U$ means that for any $a \in U$, there is $\epsilon > 0$, such that $d(x, a) < \epsilon$ implies $x \in U$.

To get some feeling toward the concept, let us consider the two subsets of the plane $\mathbb{R}^2$ in Figure 2.3.2. The subset $A$ does not contain the boundary. Any point inside $A$ has a ball around it that is contained in $A$. Although the ball has to become smaller when the point gets closer to the boundary, the important thing here is that such a ball can always be found. Therefore $A$ is open. In contrast, $B$ contains the boundary. Around a boundary point, no matter how small the ball is,

[3]Felix Hausdorff, born November 8, 1869 in Breslau, Germany (now Wrocław, Poland), died January 26, 1942 in Bonn, Germany. Hausdorff made important contributions to topology and set theory. His famous 1914 book "Grundzúge der Mengenlehre" laid down the modern foundation of topology and metric spaces. Hausdorff also published philosophical and literary work under the pseudonym of Paul Mongré.

it always contains some points outside $B$. So no ball around the point is contained in $B$, and $B$ is not open.

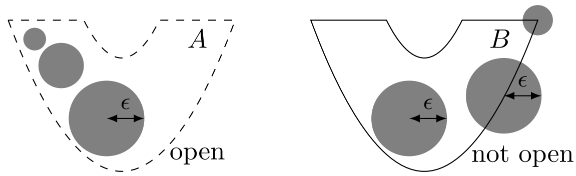



**Figure 2.3.2.** *open and not open subsets*

**Example 2.3.1.** For any subset $A$ in a discrete metric space, we have

$$a \in A \implies B(a, 1) = \{a\} \subset A.$$

This shows that any subset in a discrete metric space is open.

**Example 2.3.2.** The subset $(a, b) = \{x \in \mathbb{R} \colon a < x < b\}$ is called an open interval because it is indeed open in $\mathbb{R}_{\text{usual}}$ (meaning the real line with the usual distance $d(x, y) = |x - y|$). In fact, for any $x \in (a, b)$, we have $B(x, \epsilon) = (x - \epsilon, x + \epsilon) \subset (a, b)$ by choosing $\epsilon = \min\{x - a, b - x\}$.

The single point subset $\{a\} \subset \mathbb{R}_{\text{usual}}$ is not open. However, according to Example 2.3.1, $\{a\}$ is open in the discrete metric. This shows *openness depends on the choice of the metric.*

The closed interval $[0, 1]$ is not open in $\mathbb{R}_{\text{usual}}$ because no matter how small $\epsilon$ is, the ball $B(0, \epsilon) = (-\epsilon, \epsilon)$ around $0 \in [0, 1]$ is not contained in $[0, 1]$. Intuitively, the situation is similar to the subset $B$ in Figure 2.3.2 because 0 is a "boundary point" of $[0, 1]$.

**Example 2.3.3.** Consider the closed interval $[0, 1]$ as a subset of $X = [0, 1] \cup [2, 3]$ equipped with the usual metric. The balls in $X$ are

$$B_X(a, \epsilon) = \{x \in X \colon |x - a| < \epsilon\} = ([0, 1] \cup [2, 3]) \cap (a - \epsilon, a + \epsilon).$$

In particular, $B_X(a, 1) \subset [0, 1]$ for any $a \in [0, 1]$. Therefore $[0, 1]$ is an open subset of $X$. This shows *openness also depends on the choice of the ambient space.*

**Example 2.3.4.** The subset

$$U = \{f \in C[0, 1] \colon f(t) > 0 \text{ for all } 0 \le t \le 1\}$$

consists of all positive continuous functions. To prove that $U$ is $L_\infty$-open, for any positive function $f$, we need to find some $\epsilon > 0$, such that all functions in the ball $B_{L_\infty}(f, \epsilon)$ are still positive. Specifically, the continuous function $f$ has positive lower bound $\epsilon > 0$ on the bounded closed interval $[0, 1]$. Suppose $g \in B_{L_\infty}(f, \epsilon)$. Then $|g(t) - f(t)| < \epsilon$ for all $0 \le t \le 1$. This implies $g(t) > f(t) - \epsilon \ge 0$, so that $g \in U$.

The subset $U$ is not $L_1$-open. To see this, we take the constant function $1 \in U$ and for any $\epsilon > 0$, construct a function $f$ satisfying $d_{L_1}(f, 1) = \displaystyle\int_0^1 |f(t) - 1| dt < \epsilon$ and $f(t) \le 0$ somewhere (so that $f \notin U$). Such a function can be easily constructed by connecting a straight line from $f(0) = 0$ to $f(\epsilon) = 1$ and then connecting another straight line from $f(\epsilon) = 1$ to $f(1) = 1$. The formula for the function is

$$f(t) = \begin{cases} \epsilon^{-1}t, & \text{if } 0 \le t \le \epsilon, \\ 1, & \text{if } \epsilon \le t \le 1. \end{cases}$$

The following result provides lots of examples of open subsets.

**Lemma 2.3.2.** *Any ball $B(a, \epsilon)$ is open.*

*Proof.* For $x \in B(a, \epsilon)$, we need to find $\delta > 0$, such that $B(x, \delta) \subset B(a, \epsilon)$. Figure 2.3.3 suggests that any $\delta$ satisfying $0 < \delta \le \epsilon - d(x, a)$ should be sufficient.

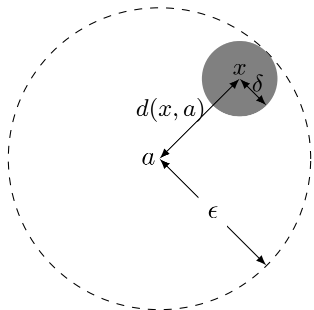



**Figure 2.3.3.** *Balls are open.*

Of course the intuition needs to be rigorously proved. The assumption $x \in B(a, \epsilon)$ implies $d(x, a) < \epsilon$, so that $\delta = \epsilon - d(x, a) > 0$ can be chosen. Then for $y \in B(x, \delta)$, by the triangle inequality, we have

$$d(y, a) \le d(y, x) + d(x, a) < \delta + d(x, a) = \epsilon.$$

This means $y \in B(a, \epsilon)$ and completes the verification that $B(x, \delta) \subset B(a, \epsilon)$. $\square$

As an application of the lemma, the interval $(a, b)$ is open because it is a ball centered at the middle point $\dfrac{a+b}{2}$ and with radius $\dfrac{b-a}{2}$.

Exercise 2.3.1. Prove that any subset in a finite metric space is open.

Exercise 2.3.2. Which are open subsets of $\mathbb{R}^2$ in the Euclidean metric?

1. $\{(x, y) \colon x < 0\}$.
2. $\{(x, y) \colon 2 < x + y < 6\}$.
3. $\{(x, y) \colon x^2 + y^2 < 1 \text{ or } (x, y) = (1, 0)\}$.
4. $\{(x, y) \colon x > 2 \text{ and } y \le 3\}$.

Exercise 2.3.3. Determine whether the interval $(0, 1]$ is open as a subset of the following spaces equipped with the usual metric $d(x, y) = |x - y|$.

1. $\mathbb{R}$.
2. $(0, \infty)$.
3. $(-\infty, 1]$.
4. $(0, 1]$.
5. $[0, 1]$.
6. $\{-1\} \cup (0, 1]$.

Exercise 2.3.4. What are the open subsets of the space $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ in the usual metric?

Exercise 2.3.5. Which one describes an open subset of $C[0, 1]$ in the $L_\infty$-metric? How about the $L_1$-metric?

1. $f(0) > 1$.
2. $f(0) \geq 1$.
3. $f(0) \geq f(1)$.
4. $t + 1 > f(t) > t^2$ for all $t$.
5. $t + 1 \geq f(t) \geq t^2$ for all $t$.
6. $\displaystyle\int_0^1 f(t)dt > 0$.
7. $\displaystyle\int_0^1 f(t)dt \geq 0$.
8. $\displaystyle\int_0^{\frac{1}{2}} f(t)dt \geq \int_{\frac{1}{2}}^1 f(t)dt$.

The functions $a(t), b(t) \in C[0, 1]$ are fixed in the fourth problem.

Exercise 2.3.6. Which one describes an open subset of $\mathbb{Q}$ in the 2-adic metric?

1. integers.
2. even integers.
3. $\dfrac{\text{odd integers}}{\text{even integers}}$.
4. $\dfrac{\text{even integers}}{\text{odd integers}}$.
5. $\dfrac{1}{\text{integers}}$.
6. $r \in \mathbb{Q}$, $|r| < 1$.

Exercise 2.3.7. Each of the following condition describes a collection of subsets $A \subset \mathbb{N}$. Which collection is an open subset of $\mathcal{P}(\mathbb{N})$ in the metric in Exercise 2.1.15?

1. All numbers in $A$ are bigger than 10.
2. All numbers in $A$ are less than 10.
3. All numbers in $A$ are even.
4. $A$ does not contain 100.
5. $A$ contains infinitely many numbers.
6. $A$ contains exactly two numbers.

Exercise 2.3.8. Prove that the subset $\{x\colon d(a, x) > \epsilon\}$ is open for any $a$ and $\epsilon \geq 0$.

Exercise 2.3.9. Suppose $d$ and $d'$ are two metrics on $X$ satisfying $d'(x, y) \leq c\, d(x, y)$ for some constant $c > 0$. Prove that $d'$-open subsets are $d$-open. In particular, if two metrics are *equivalent*:

$$c_1\, d(x, y) \leq d'(x, y) \leq c_2\, d(x, y) \text{ for some constants } c_1, c_2 > 0,$$

then $d$-open is equivalent to $d'$-open. Two metrics are said to *induce the same topology* if the openness in one metric is the same as the openness in the other.

Exercise 2.3.10. Prove that the metrics in Example 2.1.3 induce the same topology.

Exercise 2.3.11. Prove that in $C[0, 1]$, $L_1$-open implies $L_\infty$-open, but the converse is not true.

Exercise 2.3.12. Suppose $d$ and $d'$ are two metrics on $X$. Prove the following are equivalent.

1. $d'$-open subsets are $d$-open.
2. For any $x \in X$ and $\epsilon > 0$, there is $\delta > 0$, such that $d(x, y) < \delta$ implies $d'(x, y) < \epsilon$.
3. $d + d'$ and $d$ induce the same topology.

Exercise 2.3.13. Let $d_X$ and $d_Y$ be metrics on $X$ and $Y$.

1. Prove that $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ is a metric on $X \times Y$.
2. Prove that for any $p \geq 1$, $d((x_1, y_1), (x_2, y_2)) = \sqrt[p]{d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p}$ is also a metric on $X \times Y$, and induce the same topology as $d_{X\times Y}$.
3. Prove that if $U \subset X$ and $V \subset Y$ are open, then $U \times V$ is open with respect to the product metric.

The following result contains the most important properties of open subsets. The properties will become the axioms for the concept of topology.

**Proposition 2.3.3.** *The open subsets of a metric space $X$ satisfy the following:*

1. *$\emptyset$ and $X$ are open.*
2. *Unions of open subsets are open.*
3. *Finite intersections of open subsets are open.*

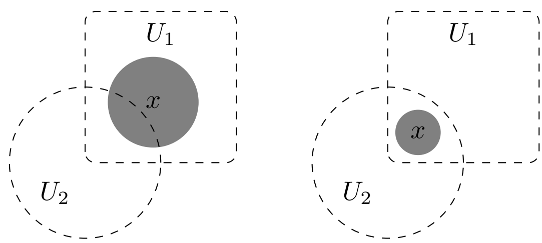



**Figure 2.3.4.** *union and intersection of open subsets*

*Proof.* The first property is trivial.

Let $U_i$ be open subsets of $X$, $i$ being some indices (this is the *assumption*). The second property says that the union $U = \cup U_i$ is also open. In other words, we need to show (this is the *conclusion*)

$$x \in U \implies \text{a ball } B(x, \epsilon) \subset U.$$

The situation is described on the left of Figure 2.3.4. Keeping in mind the assumption and the conclusion, the second property is proved as follows.

$$\begin{aligned} x \in U &\implies x \in U_i \text{ for some index } i && \text{(def of union)} \\ &\implies B(x,\epsilon) \subset U_i \text{ for some } \epsilon > 0 && \text{(def of open)} \\ &\implies B(x,\epsilon) \subset U. && (U \text{ contains } U_i) \end{aligned}$$

For the third property, it is sufficient to consider the intersection of two open subsets. Let $U_1$ and $U_2$ be open (this is the *assumption*). Then we need to argue that (this is the *conclusion*)

$$x \in U = U_1 \cap U_2 \implies \text{a ball } B(x,\epsilon) \subset U.$$

The situation is described on the right of Figure 2.3.4 and proved as follows.

$$\begin{aligned} x \in U &\implies x \in U_1, x \in U_2 && \text{(def of intersection)} \\ &\implies B(x,\epsilon_1) \subset U_1, B(x,\epsilon_2) \subset U_2 \text{ for some } \epsilon_1, \epsilon_2 > 0 && \text{(def of open)} \\ &\implies B(x,\epsilon) \subset B(x,\epsilon_1) \cap B(x,\epsilon_2) \subset U_1 \cap U_2 = U. \end{aligned}$$

In the third implication, $\epsilon$ can be any number satisfying $0 < \epsilon \le \min\{\epsilon_1, \epsilon_2\}$. □

Infinite intersections of open subsets may not be open. A counterexample is given by $\cap \left(-\dfrac{1}{n}, \dfrac{1}{n}\right) = \{0\}$ in $\mathbb{R}_{\text{usual}}$.

By Lemma 2.3.2 and the second part of Proposition 2.3.3, unions of balls are open. The following result says the converse is also true.

**Lemma 2.3.4.** *A subset of a metric space is open if and only if it is a union of balls.*

*Proof.* Suppose $U$ is open. Then for any $x \in U$, there is $\epsilon_x > 0$, such that $B(x,\epsilon_x) \subset U$. Here the subscript in $\epsilon_x$ is used to indicate the possible dependency of $\epsilon$ on $x$. After constructing one such ball for each $x \in U$, we have $\cup_{x\in U} B(x,\epsilon_x) \subset U$. On the other hand, we also have $U = \cup_{x\in U}\{x\} \subset \cup_{x\in U} B(x,\epsilon_x)$. Therefore

$$U = \cup_{x\in U} B(x,\epsilon_x),$$

which expresses $U$ as a union of balls. □

The following gives the complete description of open subsets in $\mathbb{R}_{\text{usual}}$.

**Theorem 2.3.5.** *A subset of $\mathbb{R}$ is open in the usual metric if and only if it is a disjoint union of open intervals.*

*Proof.* By the second property of Proposition 2.3.3, unions of open intervals are open. We need to prove the converse.

Let $U$ be an open subset of $\mathbb{R}$. We call $(a,b) \subset U$ a maximal interval if $(a,b) \subset (c,d) \subset U$ implies $(a,b) = (c,d)$. If two maximal intervals $(a_1,b_1)$ and $(a_2,b_2)$ intersect, then $(a_1,b_1) \cup (a_2,b_2) = (a,b)$, with $a = \min\{a_1,a_2\}$ and $b = \max\{b_1,b_2\}$, is an interval contained in $U$. By the maximal property of $(a_1,b_1)$ and $(a_2,b_2)$, we get $(a_1,b_1) = (a,b) = (a_2,b_2)$. Therefore distinct maximal intervals are disjoint.

By the definition, the union of all maximal intervals is contained in $U$. Conversely, we need to prove that any $x \in U$ is inside a maximal interval. Since $U$ is open, we have $x \in (a,b) \subset U$ for some open interval $(a,b)$. Therefore we can define

$$a_x = \inf_{x\in(a,b)\subset U} a, \quad b_x = \sup_{x\in(a,b)\subset U} b.$$

We expect $(a_x,b_x)$ to be the maximal interval containing $x$.

First we show $(a_x,b_x) \subset U$. For any $y \in (a_x,b_x)$, we have either $a_x < y \le x$ or $x \le y < b_x$. If $a_x < y \le x$, then by the definition of $a_x$, there is an interval $(a,b)$, such that $x \in (a,b) \subset U$ and $a_x < a < y$. Further by $y \le x < b$, we get $y \in (a,b) \subset U$. It can be similarly proved that $x \le y < b_x$ also implies $y \in U$. This proves that $(a_x,b_x) \subset U$.

It remains to show that $(a_x,b_x)$ is maximal. Suppose $(a_x,b_x) \subset (c,d) \subset U$. Then $c \le a_x$. On the other hand, by $x \in (a_x,b_x)$, we get $x \in (c,d) \subset U$. Then by the definition of $a_x$, we have $a_x \le c$. Thus we conclude that $c = a_x$. Similarly, we can prove $d = b_x$. This shows that $(a_x,b_x)$ is maximal. $\square$

Exercise 2.3.14. Prove that the complement of any single point in a metric space is an open subset. What about the complement of two points?

Exercise 2.3.15. For a subset $A$ of a metric space $X$ and $\epsilon > 0$, define the *$\epsilon$-neighborhood* of $A$ to be

$$A^\epsilon = \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A\}.$$

For example $\{a\}^\epsilon = B(a,\epsilon)$. Prove that $A^\epsilon$ is an open subset of $X$.

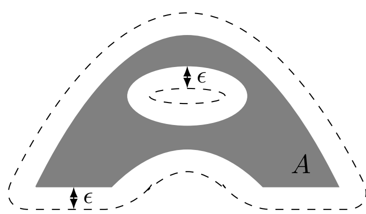



**Figure 2.3.5.** *$\epsilon$-neighborhood of $A$*

## 2.4 Continuity

As promised before, the concept of continuity can be defined for maps between metric spaces.

**Definition 2.4.1.** A map $f: X \to Y$ between metric spaces is *continuous* if for any $a \in X$ and $\epsilon > 0$, there is $\delta > 0$, such that

$$d_X(x,a) < \delta \implies d_Y(f(x), f(a)) < \epsilon.$$

**Example 2.4.1.** Let $X$ be a discrete metric space and let $Y$ be any metric space. Then any map $f: X \to Y$ is continuous. To see this, for any given $a$ and $\epsilon$, take $\delta = 1$. Because $X$ is discrete, $d(x,a) < \delta = 1$ implies $x = a$, so that $d(f(x), f(a)) = 0 < \epsilon$.

**Example 2.4.2.** The evaluation map $E(f) = f(0): C[0,1]_{L_\infty} \to \mathbb{R}_{\text{usual}}$ is continuous. In fact, we have

$$d(E(f), E(g)) = |f(0) - g(0)| \le \max_{0 \le t \le 1} |f(t) - g(t)| = d_\infty(f, g).$$

Thus by choosing $\delta = \epsilon$,

$$d_\infty(f,g) < \delta \implies d(E(f), E(g)) \le \delta = \epsilon.$$

Example 2.4.8 will show that $E$ is no longer continuous if the $L_\infty$-metric is changed to the $L_1$-metric.

**Example 2.4.3.** Let $f: X \to Y$ and $g: X \to Z$ be continuous maps. Consider $h(x) = (f(x), g(x)): X \to Y \times Z$, where $Y \times Z$ has the product metric in Exercise 2.3.13.

For any $a \in X$ and $\epsilon > 0$, there are $\delta_1, \delta_2 > 0$, such that

$$\begin{aligned} d_X(x,a) < \delta_1 &\implies d_Y(f(x), f(a)) < \epsilon, \\ d_X(x,a) < \delta_2 &\implies d_Z(g(x), g(a)) < \epsilon. \end{aligned}$$

Then for $\delta = \min\{\delta_1, \delta_2\}$, we have

$$d_X(x,a) < \delta \implies d_{Y\times Z}(h(x), h(a)) = \max\{d_Y(f(x), f(a)), d_Z(g(x), g(a))\} < \epsilon.$$

This shows that $h$ is continuous.

**Example 2.4.4.** The addition map $\sigma(x,y) = x+y: \mathbb{R}^2 \to \mathbb{R}$ is well known to be continuous with respect to the Euclidean metrics. Now consider the addition map $\sigma: \mathbb{Q}^2_{p\text{-adic}} \to \mathbb{Q}_{p\text{-adic}}$ with respect to the $p$-adic metric, where the metric on $\mathbb{Q}^2$ is given in Exercise 2.3.13. We claim that

$$d_{p\text{-adic}}(x_1+y_1, x_2+y_2) \le d_{p\text{-adic}}((x_1,y_1),(x_2,y_2)) = \max\{d_{p\text{-adic}}(x_1,x_2), d_{p\text{-adic}}(y_1,y_2)\}.$$

The inequality implies that the condition for the continuity of the addition map holds for $\delta = \epsilon$, so that the addition map is continuous.

The inequality can be proved as follows. Let

$$\begin{aligned} x_1 - x_2 &= \frac{m}{n}p^a, & d_{p\text{-adic}}(x_1,x_2) &= p^{-a}, \\ y_1 - y_2 &= \frac{k}{l}p^b, & d_{p\text{-adic}}(y_1,y_2) &= p^{-b}. \end{aligned}$$

Then by the similar argument as in Example 2.1.5, there is an integer $c \ge 0$, such that

$$d_{p\text{-adic}}(x_1+y_1, x_2+y_2) = p^{-(c+\min\{a,b\})} \le \max\{p^{-a}, p^{-b}\}.$$

Exercise 2.4.1. Prove that the definition of continuity is not changed if $<$ in the definition is replaced by $\leq$?

Exercise 2.4.2. Determine the continuity for the $L_1$- and $L_\infty$-metrics.

1. $M(f) = \max_{[0,1]} f\colon C[0,1] \to \mathbb{R}_{\text{usual}}$.
2. $\mathcal{I}(f) = \displaystyle\int_0^1 f(t)dt\colon C[0,1] \to \mathbb{R}_{\text{usual}}$.
3. $\sigma(f,g) = f + g\colon C[0,1]^2 \to C[0,1]$.

Exercise 2.4.3. Prove that the identity map $id\colon C[0,1]_{L_\infty} \to C[0,1]_{L_1}$ is continuous. What about the other identity map $id\colon C[0,1]_{L_1} \to C[0,1]_{L_\infty}$?

Exercise 2.4.4. Is the multiplication map $\mu(x,y) = xy\colon \mathbb{Q}^2_{p\text{-adic}} \to \mathbb{Q}_{p\text{-adic}}$ continuous in the $p$-adic metric?

Exercise 2.4.5. Consider the metric space in Exercise 2.1.15.

1. Prove that $d(A\cup E, B\cup E) \leq d(A,B)$. Then use this to prove that for a fixed subset $E \subset \mathbb{N}$, the map $A \mapsto A \cup E$ is continuous from $\mathcal{P}(\mathbb{N})$ to itself. What about the maps $A \mapsto A \cap E$, $A \mapsto A - E$ and $A \mapsto E - A$?
2. Prove that $f(A) = \{2a\colon a \in A\}$ and $g(A) = \{a\colon 2a \in A\}$ are continuous.
3. The union, intersection and difference can be considered as maps from $\mathcal{P}(\mathbb{N})^2$ to $\mathcal{P}(\mathbb{N})$. Are these maps continuous?
4. Prove that
$$h(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ \dfrac{1}{\min A}, & \text{if } A \neq \emptyset \end{cases} : \mathcal{P}(\mathbb{N}) \to \mathbb{Q}_{\text{usual}}$$
is continuous. What if the metric on $\mathbb{Q}$ is the $p$-adic one?

Exercise 2.4.6. Prove that if $f, g\colon X \to \mathbb{R}_{\text{usual}}$ are continuous, then $f + g$ and $fg$ are also continuous.

Exercise 2.4.7. Prove that if $f\colon X \to Y$ and $g\colon Z \to W$ are continuous, then the map $h(x,z) = (f(x), g(z))\colon X \times Z \to Y \times W$ is continuous.

Exercise 2.4.8. Prove that if $f\colon X \to Y$ satisfies $d_Y(f(x), f(y)) \leq c\, d_X(x,y)$ for a constant $c > 0$, then $f$ is continuous.

Exercise 2.4.9. Let $f\colon X \to Y$ be a continuous map. Is the map still continuous if the metric $d_X$ on $X$ is changed to another metric $d'_X$ satisfying $d'_X(x_1,x_2) \leq c\, d_X(x_1,x_2)$ for a constant $c > 0$? What if $d'_X$ satisfies $d_X(x_1,x_2) \leq c\, d'_X(x_1,x_2)$ instead? What if the metric on $Y$ is also similarly modified?

Exercise 2.4.10. Fix a point $a$ in a metric space $X$. Prove that $f(x) = d(x,a)\colon X \to \mathbb{R}_{\text{usual}}$ is continuous.

Exercise 2.4.11. Let $X$ be a metric space with distance $d$. Let $X \times X$ have the metric given in Exercise 2.3.13. Prove that $d\colon X \times X \to \mathbb{R}_{\text{usual}}$ is a continuous map.

Exercise 2.4.12. A sequence $\{a_n : n \in \mathbb{N}\}$ in a metric space $X$ is said to have *limit* $x$ (or *converges* to $x$) and denoted $\lim a_n = x$, if for any $\epsilon > 0$, there is $N$, such that $n > N$ implies $d(a_n, x) < \epsilon$. Prove that $f\colon X \to Y$ is continuous if and only if $\lim a_n = x$ implies $\lim f(a_n) = f(x)$.

In terms of balls, the implication in the definition of continuity means

$$f(B(a,\delta)) \subset B(f(a),\epsilon).$$

This is further equivalent to

$$B(a,\delta) \subset f^{-1}(B(f(a),\epsilon)).$$

Such set-theoretical interpretation leads to the interpretation of the continuity in terms of open subsets.

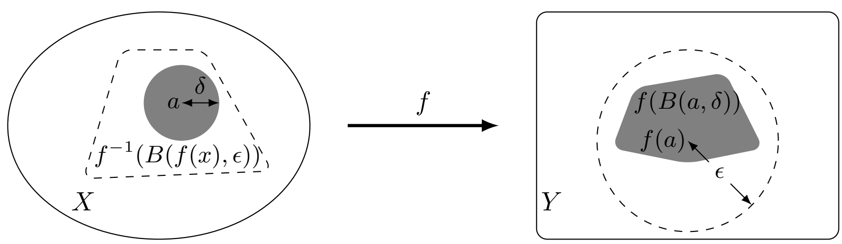



**Figure 2.4.1.** *continuity in terms of balls*

**Proposition 2.4.2.** *The following are equivalent for a map $f\colon X \to Y$ between metric spaces.*

1. *The map $f$ is continuous.*
2. *The preimage $f^{-1}(U)$ of any open subset $U \subset Y$ is open in $X$.*
3. *The preimage $f^{-1}(B(y,\epsilon))$ of any ball in $Y$ is open in $X$.*

*Proof.* Suppose $f\colon X \to Y$ is continuous. Let $U \subset Y$ be an open subset. The following proves that $f^{-1}(U)$ is open.

$$\begin{aligned} a \in f^{-1}(U) &\implies f(a) \in U \\ &\implies B(f(a),\epsilon) \subset U \text{ for some } \epsilon > 0 \\ &\implies B(a,\delta) \subset f^{-1}(B(f(a),\epsilon)) \subset f^{-1}(U) \text{ for some } \delta > 0. \end{aligned}$$

In the last step, the rephrase of the continuity is used. Therefore the first property implies the second.

By Lemma 2.3.2, the second property implies the third.

It remains to prove that the third property implies the first. Suppose $f$ has the third property. Then for any $a \in X$ and $\epsilon > 0$, the preimage $f^{-1}(B(f(a), \epsilon))$ is open. Since $a \in f^{-1}(B(f(a), \epsilon))$, there is $\delta > 0$, such that $B(a, \delta) \subset f^{-1}(B(f(a), \epsilon))$. As explained before, this is equivalent to the continuity of $f$. □

**Corollary 2.4.3.** *Compositions of continuous maps are continuous.*

*Proof.* Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous. Let $U \subset Z$ be open. By Proposition 2.4.2, it is sufficient to verify that $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open. By the proposition, the continuity of $g$ implies that $g^{-1}(U)$ is open, and the continuity of $f$ further implies that $f^{-1}(g^{-1}(U))$ is open. □

**Example 2.4.5.** To determine whether the addition map $\sigma(x, y) = x + y\colon \mathbb{R}^2_{\text{Euclidean}} \to \mathbb{R}_{\text{usual}}$ is continuous, only the preimages $\sigma^{-1}(a, b)$ of open intervals need to be considered. The preimages are open strips in $\mathbb{R}^2$ at 135 degrees, and it is intuitively clear (and not difficult to show) that they are indeed open. Therefore $\sigma$ is continuous.

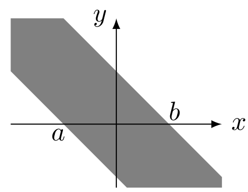



**Figure 2.4.2.** $\sigma^{-1}(a, b)$

Note that the second property in Proposition 2.4.2 indicates that the continuity depends only on open subsets. Since open in the Euclidean metric is the same as open in the taxicab metric (see Exercise 2.3.10), $\sigma$ is still continuous if $\mathbb{R}^2$ is given the taxicab metric.

**Example 2.4.6.** The subset of triples $(x, y, z)$ satisfying the equation $1 < x^2 + y^2 + z^2 + xy + yz + zx < 2$ is an open subset of $\mathbb{R}^3_{\text{Euclidean}}$. This can be seen by considering the continuous map $f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + zx\colon \mathbb{R}^3 \to \mathbb{R}_{\text{usual}}$. Then the subset is simply the preimage of the open interval $(1, 2)$ under the continuous map. By Proposition 2.4.2, the subset is open.

In general, subsets of Euclidean spaces described by finitely many strict inequalities between continuous functions are open in the Euclidean metric.

**Example 2.4.7.** The set $M(n, n)$ of all $n \times n$ matrices may be identified with $\mathbb{R}^{n^2}$ in an obvious way. The determinant map $\det\colon M(n, n) \to \mathbb{R}$ is a big multivariable polynomial, and is continuous with respect to the usual metrics. Therefore the preimage $\det^{-1}(\mathbb{R} - \{0\})$, consisting of all invertible matrices, is an open subset.

**Example 2.4.8.** Example 2.4.2 showed that the evaluation map $E(f) = f(0)\colon C[0, 1]_{L_\infty} \to \mathbb{R}_{\text{usual}}$ is continuous. Therefore the subset $E^{-1}(1, \infty) = \{f \in C[0, 1]\colon f(0) > 1\}$ is $L_\infty$-

open. More generally, by the same argument as in Example 2.4.2, the evaluation $E_t(f) = f(t)\colon C[0,1]_{L_\infty} \to \mathbb{R}_{\text{usual}}$ at (fixed) $t$ is also continuous. Therefore for fixed $a, t, \epsilon$, the subset

$$B(a,t,\epsilon) = \{f \in C[0,1]\colon |f(t) - a| < \epsilon\} = E_t^{-1}(a-\epsilon, a+\epsilon)$$

is $L_\infty$-open.

On the other hand, the subset $E^{-1}(1,\infty)$ is not $L_1$-open. We have the constant function $2 \in E^{-1}(1,\infty)$. But the function (compare with the construction in Example 2.1.13)

$$f(t) = \begin{cases} 2\epsilon^{-1}t, & \text{if } 0 \le t \le \epsilon, \\ 2, & \text{if } \epsilon \le t \le 1, \end{cases}$$

satisfies $f \notin E^{-1}(1,\infty)$ and $f \in B_{L_1}(2, 2\epsilon)$. Therefore the evaluation map $E\colon C[0,1]_{L_1} \to \mathbb{R}_{\text{usual}}$ is not continuous.

Exercise 2.4.13. Draw the pictures of the preimages of open intervals under the subtraction map, the multiplication map, and the division map. Then explain why these maps are continuous.

Exercise 2.4.14. By Example 2.4.1, any map *from* a discrete metric space is continuous. How about maps *to* a discrete metric space? The answer gives us examples of *discontinuous* maps such that the *images* of open subsets are open.

Exercise 2.4.15. Let $A$ be a subset of a metric space $X$. Consider the distance function from $A$

$$d(x,A) = \inf\{d(x,a)\colon a \in A\}\colon X \to \mathbb{R}.$$

1. Prove $|d(x,A) - d(y,A)| \le d(x,y)$.
2. Prove $d(x,A)$ is continuous in $x$.
3. Use the continuity of $d(x,A)$ to give another proof of Exercise 2.3.15.

## 2.5 Limit Point

**Definition 2.5.1.** A point $x \in X$ is a *limit point* of $A \subset X$ if for any $\epsilon > 0$, there is a point $a$ satisfying

$$a \in A, \quad a \ne x, \quad d(a,x) < \epsilon.$$

The collection of limit points of $A$ is denoted $A'$.

So $x$ is a limit point of $A$ if there are points of $A$ that are arbitrarily close but not equal to $x$. The condition is equivalent to

$$(A - x) \cap B(x,\epsilon) \ne \emptyset$$

for any $\epsilon > 0$. In calculus, the concept of limit points are introduced in terms of convergent sequences. See Exercise 2.5.5 for such relation in general.

**Example 2.5.1.** By taking $\epsilon = 1$, it is easy to see that any subset of a discrete metric space has no limit point.

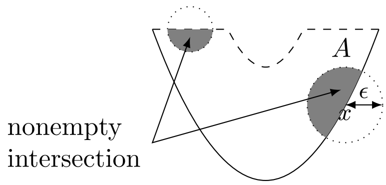



**Figure 2.5.1.** *$x$ is a limit point of $A$.*

**Example 2.5.2.** The limit points of $A_1 = [0,1]\cup\{2\}$, $A_2 = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$, $A_3 = \mathbb{Q}$, $A_4 = \mathbb{Z}$ in $\mathbb{R}_{\text{usual}}$ are respectively $A_1' = [0,1]$, $A_2' = \{0\}$, $A_3' = \mathbb{R}$, $A_4' = \emptyset$.

The point $x = 2$ is not a limit point of $A_1$ because the only $a \in A_1$ satisfying $d(a,x) < 1$ is $a = x$. Therefore the three conditions cannot be satisfied at the same time. Intuitively, the reason for 2 not to be a limit point is because it is an *isolated point* of $A_1$, in the sense that there are no other points of $A_1$ nearby. The same intuition applies to $A_4$, in which all the points are isolated, so that no points are limit points.

**Example 2.5.3.** The *closed ball* is $\bar{B}(a,\epsilon) = \{x \colon d(x,a) \le \epsilon\}$. We have $B(a,\epsilon)' = \bar{B}(a,\epsilon)$ in the Euclidean space $\mathbb{R}^n$. Example 2.5.1 shows that the equality may not hold in general metric spaces.

**Example 2.5.4.** Is the constant zero function 0 (taken as $x$ in the definition) a limit point of the subset $A = \{f \colon f(1) = 1\} \subset C[0,1]$?

For the $L_\infty$-metric, the problem is the existence of a function $f$ (taken as $a$ in the definition) satisfying $f(1) = 1$, $f \ne 0$, and $|f(t)| < \epsilon$ for all $0 \le t \le 1$. Since the conditions are contradictory for $\epsilon = 1$, the constant zero function is not an $L_\infty$-limit point of $A$.

For the $L_1$-metric, the problem is the existence of $f$ satisfying $f(1) = 1$, $f \ne 0$, and $\int_0^1 |f(t)|dt < \epsilon$. No matter how small $\epsilon$ is, there is a big $N$ ($N > \epsilon^{-1}$ is good enough), such that $f(t) = t^N$ satisfies the three conditions. Therefore 0 is an $L_1$-limit point of $A$.

**Example 2.5.5.** A fundamental theorem in approximation theory says that any continuous function on $[0,1]$ may be uniformly approximated by polynomials. Specifically, this says that for any continuous function $f(t)$ and $\epsilon > 0$, there is a polynomial $p(t)$, such that $|f(t) - p(t)| < \epsilon$ for all $0 \le t \le 1$. Therefore the $L_\infty$-limit points of the subset of polynomials is the whole space $C[0,1]$.

Exercise 2.5.1. Find the limit points of subsets in Exercises 2.3.5, 2.3.6, 2.3.7.

Exercise 2.5.2. Suppose there is $\epsilon > 0$, such that $d(x,y) > \epsilon$ for any $x, y \in A$. Prove that $A$ has no limit point. This implies that any finite subset of a metric space has no limit point.

Exercise 2.5.3. Let $x$ be a limit point of $A$. Is $x$ still a limit point of $A$ if the metric is modified as in Exercise 2.3.9?

Exercise 2.5.4. Prove properties of limit points.

1. $A \subset B \implies A' \subset B'$.
2. $(A \cup B)' = A' \cup B'$.
3. $(A \cap B)' \subset A' \cap B'$, and the two sides may not be equal.
4. $A'' \subset A'$, and the two sides may not be equal.

Moreover, show that $(X - A)'$ may not be equal to $X - A'$.

Exercise 2.5.5. Recall the definition of the limit of sequences in Exercise 2.4.12. A sequence $a_n$ is *non-repetitive* if $a_i \neq a_j$ for $i \neq j$. Prove that $x$ is a limit point of $A$ if and only if it is the limit of a non-repetitive sequence in $A$.

Exercise 2.5.6. Construct examples to show that $A'$ can be bigger than, smaller than, or equal to $A$?

Exercise 2.5.7. Prove that $B(a, \epsilon)' \subset \bar{B}(a, \epsilon)$ in general, and the equality holds in $C[0,1]_{L_1}$ and $C[0,1]_{L_\infty}$.

Exercise 2.5.8. If $x$ is a limit point of a subset $A$. Let $U$ be an open subset containing $x$. Prove that $x$ is a limit point of $A \cap U$, and $A \cap U$ contains infinitely many points.

Exercise 2.5.9. If $x$ is a limit point of $A \subset X$ and $f : X \to Y$ is a continuous map, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Exercise 2.5.10. Let $A$ be a subset of a metric space. Let $d(x, A)$ be the distance function defined in Exercise 2.4.14. Prove that $d(x, A) = 0$ if and only if $x \in A$ or $x \in A'$.

## 2.6 Closed Subset

The notion of closed subsets is motivated by the intuition that nothing "escapes" from them.

**Definition 2.6.1.** A subset of a metric space is *closed* if it contains all its limit points.

By Example 2.5.1, any subset of a discrete metric space is closed. In Example 2.5.2, only $A_1'$ and $A_4'$ are contained respectively in $A_1$ and $A_4$. Therefore $A_1$ and $A_4$ are closed, while $A_2$ and $A_3$ are not closed.

Example 2.5.3 implies that points of distance $\epsilon$ from $a$ are limit points of the Euclidean ball $B(a, \epsilon)$. Since such points are not in $B(a, \epsilon)$, the Euclidean ball is not closed.

Example 2.5.5 tells us that the subset of all polynomials is not closed in $C[0,1]_{L_\infty}$. In fact, it is also not closed in $C[0,1]_{L_1}$.

Finally, by Exercise 2.5.2, any finite subset of a metric space is closed.

Exercise 2.6.1. There are four possibilities for a subset $A$ of a metric space $X$:

1. $A$ is open and closed.
2. $A$ is open but not closed.
3. $A$ is closed but not open.
4. $A$ is neither open nor closed.

Show that each possibility may happen with $A \neq \emptyset, X$. Moreover, prove that the only open and closed subsets of $\mathbb{R}_{\text{usual}}$ are $\emptyset$ and $\mathbb{R}$.

Exercise 2.6.2. Let $A$ be a closed subset of a metric space $X$. Is $A$ still closed if the metric is modified as in Exercise 2.3.9?

Exercise 2.6.3. Use Exercise 2.5.10 to characterize closed subsets $A$ in terms of the function $d(x, A)$.

Exercise 2.6.4. For any two bounded subsets $A, B$ in a metric space $X$ (see Exercise 2.2.7), use the definition in Example 2.4.14 to further define

$$D(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

1. Prove that $D$ is a metric on the set $\mathcal{P}_b(X)$ of all bounded and closed subsets in $X$.
2. For any fixed $E \in X$, study the continuity of the maps $A \mapsto A \cup E$, $A \mapsto A \cap E$, $A \mapsto A - E$ and $A \mapsto E - A$. Some boundedness condition may need to be assumed.
3. Study the continuity of the union, intersection, and difference as maps from $\mathcal{P}_b(X)^2$ to $\mathcal{P}_b(X)$.
4. Study the continuity of the map $d(x, A) \colon X \times \mathcal{P}_b(X) \to \mathbb{R}_{\text{usual}}$.

Exercise 2.6.5. Prove that in an ultrametric space (see Exercise 2.2.8), any ball is also closed.

Exercise 2.6.6. Prove that the closed ball $\bar{B}(a, \epsilon)$ in Example 2.5.3 is indeed a closed subset.

Exercise 2.6.7. Prove that a subset is closed if and only if it contains all the limits of convergent sequences in the subset.

The following is the key relation between open and closed subsets.

**Proposition 2.6.2.** *A subset $C \subset X$ is closed if and only if the complement $X - C$ is open.*

*Proof.*

$$\begin{aligned}
X - C \text{ is open} &\iff x \in X - C \text{ implies } B(x, \epsilon) \subset X - C \text{ for some } \epsilon > 0 \\
&\iff x \notin C \text{ implies } C \cap B(x, \epsilon) = \emptyset \text{ for some } \epsilon > 0 \\
&\iff x \notin C \text{ implies } (C - x) \cap B(x, \epsilon) = \emptyset \text{ for some } \epsilon > 0 \\
&\iff x \notin C \text{ implies } x \text{ is not a limit point of } C \\
&\iff x \text{ is a limit point of } C \text{ implies } x \in C \\
&\iff C \text{ is closed.}
\end{aligned}$$

The third equivalence comes from $C = C - x$ when $x \notin C$. □

By Example 2.3.1, any subset of a discrete space is open. Then by Proposition 2.6.2, any subset of a discrete space is also closed. Moreover, by making use of $f^{-1}(Y - C) = X - f^{-1}(C)$, Proposition 2.6.2 implies the following.

**Corollary 2.6.3.** *A map between metric spaces is continuous if and only if the preimage of any closed subset is closed.*

By making use of de Morgan's law $X - \cup A_i = \cap(X - A_i)$ and $X - \cap A_i = \cup(X - A_i)$, Propositions 2.3.3 and 2.6.2 imply the following.

**Corollary 2.6.4.** *The closed subsets of a metric space $X$ satisfy the following:*

1. *$\emptyset$ and $X$ are closed.*
2. *Intersections of closed subsets are closed.*
3. *Finite unions of closed subsets are closed.*

**Example 2.6.1.** The solutions of the equation $x^2 - y^2 = 1$ form a pair of hyperbola $H$ in $\mathbb{R}^2$. The picture suggests that $H$ is closed in the Euclidean metric. To prove this rigorously, we note that $H$ is the preimage $f^{-1}(1)$ under the map $f(x, y) = x^2 - y^2 \colon \mathbb{R}^2_{\text{Euclidean}} \to \mathbb{R}_{\text{usual}}$. Since the map is continuous and $\{1\} \subset \mathbb{R}$ is a closed subset, the preimage is closed.

The argument generalizes to higher dimensional cases, when it is often hard to argue directly from the picture that some solution set is closed. For example, the set of $2 \times 2$ singular matrices is a closed subset of $\mathbb{R}^4$ because the map

$$f(x, y, z, w) = \det \begin{pmatrix} x & y \\ z & w \end{pmatrix} = xw - yz \colon \mathbb{R}^4 \to \mathbb{R}$$

is continuous and $\{0\} \subset \mathbb{R}$ is closed.

Exercise 2.6.8. Find an example showing that the infinite union of closed subsets may not be closed.

Exercise 2.6.9. Determine whether the interval $(0, 1]$ is closed as a subset of the spaces in Exercise 2.3.3.

Exercise 2.6.10. What are the closed subsets of the space $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ in the usual metric?

Exercise 2.6.11. Which are closed subsets of $\mathbb{R}^2$ in the Euclidean metric?

1. $\{(x, y) \colon x = 0, y \leq 5\}$.
2. $\mathbb{N} \times \mathbb{Z}$.
3. $\{(x, y) \colon x^2 + y^2 < 1 \text{ or } (x, y) = (1, 0)\}$.
4. $\{(x, y) \colon y = x^2\}$.

Exercise 2.6.12. Show the following are closed in the Euclidean metric by finding suitable continuous maps.

1. $(x, y, z) \in \mathbb{R}^3$ satisfying $-1 \le x^3 - y^3 + 2z^3 - xy - 2yz + 3zx \le 3$.
2. $(x, y, z, w) \in \mathbb{R}^4$ satisfying $x^4 + y^4 = z^4 + w^4$ and $x^3 + z^3 = y^3 + z^3$.
3. All $n \times n$ orthogonal matrices.
4. All $n \times n$ matrices with all eigenvalues equal to 1.

Exercise 2.6.13. The quadratic forms

$$q(x_1, \dots, x_n) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j, \quad a_{ij} = a_{ji},$$

in $n$-variables can be identified with the Euclidean space $\mathbb{R}^{\frac{n(n+1)}{2}}$. A quadratic form $q$ is called *semi-positive definite* if $q(x_1, \dots, x_n) \ge 0$ for any vector $(x_1, \dots, x_n)$. Prove that the set of semi-positive definite quadratic forms is a closed subset in the Euclidean metric.

Exercise 2.6.14. Which subsets in Exercises 2.3.5, 2.3.6, 2.3.7 are closed?

Exercise 2.6.15. Prove that if $A$ and $B$ are closed subsets of $X$ and $Y$, then $A \times B$ is a closed subset of $X \times Y$ with respect to the product metric in Exercise 2.3.13.

Exercise 2.6.16. For disjoint closed subsets $A$ and $B$ in a metric space $X$, use Exercises 2.4.15 and 2.6.3 to construct a continuous function $f$ on $X$, such that $A = f^{-1}(0)$, $B = f^{-1}(1)$, and $0 \le f(x) \le 1$.

Exercise 2.6.17. For a subset $A$ of a metric space $X$ and $\epsilon > 0$, define the *$\epsilon$-shrinking* of $A$ to be

$$A^{-\epsilon} = \{x \in X \colon B(x, \epsilon) \subset A\}.$$

Prove that $A^{-\epsilon}$ is closed.

Chapter 3

# Graph and Network

## 3.1 Seven Bridges in Königsberg

Perhaps the first work which deserves to be considered as the beginning of topology is due to Euler[4]. In 1736 Euler published a paper entitled "Solutio problematis ad geometriam situs pertinentis" ("The solution of a problem relating to the geometry of position"). The title indicates that Euler was aware that he was dealing with a different type of geometry where distance is not relevant.

The problem studied in that paper was concerned with the geography of the city of Königsberg. The city was divided by a river, and there were two islands in the river. Seven bridges connected the two parts of the city and the two islands together.

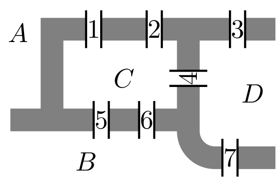



**Figure 3.1.1.** *the city of Königsberg*

Euler's asked the following question: Is it possible to cross the seven bridges in a single journey (i.e., crossing each bridge exactly once)? The problem may be abstracted into Figure 3.1.2, in which each region of the city becomes a point and each bridge becomes a line connecting two points. The abstraction of the city of Königsberg allows us to generalize Euler's problem.

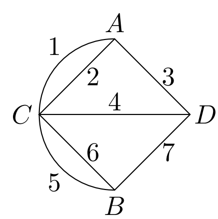



**Figure 3.1.2.** *the graph of the city of Königsberg*

**Definition 3.1.1.** A *graph* consists of finitely many points (called *vertices*), and finitely many lines (called *edges*) connecting these points. Moreover,

- A *path* is a sequence of vertices $V_1$, $V_2$, ..., $V_k$, and a sequence of edges $E_1$,

[4]Leonhard Paul Euler, born April 15, 1707 in Basel, Switzerland, died September 18, 1783 in St. Petersburg, Russia. Euler is one of the greatest mathematicians of all time. He made important discoveries in almost all areas of mathematics. Many theorems, quantities and equations are named after Euler. He also introduced much of the modern mathematical terminology and notation, including $f(x)$, $e$, $\Sigma$ (for summation), $i$ (for $\sqrt{-1}$), and modern notations for trigonometric functions.

$E_2, \ldots, E_{k-1}$, such that $E_i$ connects $V_i$ and $V_{i+1}$. The path is said to *connect* $V_1$ to $V_k$.

- A *cycle* is a path satisfying $V_1 = V_k$. In other words, the beginning point is the same as the end point.
- A graph is *connected* if any two vertices can be connected by a path.
- The *degree* of a vertex is the number of edges emanating from it.

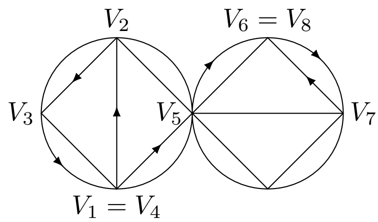



**Figure 3.1.3.** *a path in a graph*

The Königsberg graph is connected. The degrees of $A$, $B$, $D$ are 3. The degree of $C$ is 5.

Euler's problem can be generalized as follows: Given a graph, is it possible to find a path in which each edge appears exactly once?

**Theorem 3.1.2.** *The following are equivalent for a connected graph.*

1. *There is a path in which each edge appears exactly once.*
2. *There are at most two vertices with odd degrees.*

The proof is deferred to the next section and will rely on the following result.

**Proposition 3.1.3.** *In any graph,* $\sum_{\text{all vertices } V} \deg(V) = 2(\text{number of edges})$.

For the Königsberg graph, the equality means

$$\deg(A) + \deg(B) + \deg(C) + \deg(D) = 3 + 3 + 3 + 5 = 2 \times 7.$$

To see why the equality holds for the Königsberg graph, observe that the edges labeled 1, 2, 3 are counted toward $\deg(A)$. Note that the edge labeled 3 is also counted toward $\deg(C)$. In fact, the edge labeled 3 is only counted once in $\deg(A)$ and once in $\deg(C)$, and is not counted toward the degrees of the other vertices. In general, because each edge has exactly two end vertices, each of the seven edges is counted toward exactly two degrees in the summation. This is why we have twice of seven on the right side of the equation.

The key property used in the discussion above is that each edge has exactly two end vertices. Since any graph has this key property, the argument applies in general.

On closer inspection, a special case needs to be considered. A *loop* is an edge for which the two end vertices are the same $V$. Thus the loop is emanating from $V$ in two ways, and this loop should be counted twice toward $\deg(V)$. This is balanced by the right side of the formula.

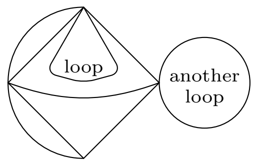



**Figure 3.1.4.** *loops*

Exercise 3.1.1. Can you find a graph with the following numbers as the degrees of the vertices?

1. 1, 2, 3.
2. 2, 2, 3.
3. 1, 2, 2, 3, 3, 4, 4.
4. 2, 2, 2, 3, 3, 3, 5.

Can you find the necessary and sufficient condition for a list of numbers to be the degrees of vertices in a graph? What about a connected graph?

Next is an interesting application of Proposition 3.1.3. The game "sprouts" starts with a number of dots on a sheet of paper. Several players take turns to draw a curve and add a new dot anywhere along the new curve. The curve must obey the following rules:

1. The curve starts and ends at existing dots (possibly the same).
2. The curve cannot cross itself.
3. The curve cannot cross any previous curves.
4. The curve cannot pass any other dot.

Another rule of the game is

5. There cannot be more than three curves emanating from any dot.

The first player who cannot add more curve loses.

**Theorem 3.1.4.** *Starting with $n$ dots, it takes at most $3n$ steps to finish the game.*

*Proof.* The game creates a graph, with dots as the vertices, and (half) curves as edges. Each step creates one more vertex and two more edges. After $N$ steps, the number of vertices is $v = n + N$, and the number of edges is $e = 2N$.

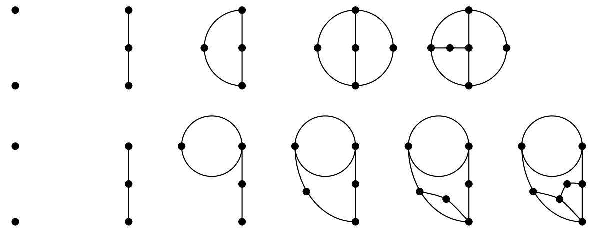


**Figure 3.1.5.** *two possible developments of "sprouts" starting from two points*

By the fifth rule, the degree of each vertex is $\leq 3$. Therefore by Proposition 3.1.3, we have $2e \leq 3v$. Substituting $v = n + N$ and $e = 2N$ into the inequality gives $4N \leq 3(n + N)$. This is the same as $N \leq 3n$. $\square$

As a matter of fact, the game has at most $3n - 1$ steps.

Exercise 3.1.2. What would be the outcome of the game if the rule is changed in one of the following ways?

1. The fifth rule is changed from three curves to four curves.
2. Two dots are added in each step instead of just one dot.
3. The game is carried out over a sphere or a torus instead of the plane.

## 3.2 Proof of One-Trip Criterion

This section is devoted to the proof of Theorem 3.1.2.

*Proof of* $1 \implies 2$. Let $\alpha$ be a path connecting a vertex $V$ to a vertex $V'$, such that each edge appears exactly once. Consider the vertices passed when we traverse along $\alpha$.

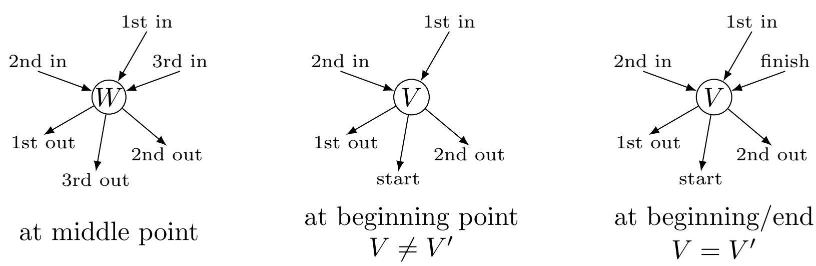



**Figure 3.2.1.** *path passing a vertex*

Whenever $\alpha$ passes a vertex $W$ different from $V$ and $V'$, the path must arrive at $W$ along one edge and leave along another edge. Thus each passage through $W$ makes use of two edges emanating from the vertex. Since each edge appears exactly once in $\alpha$, this implies

$$\deg(W) = 2(\text{number of times } \alpha \text{ passes } W).$$

In particular, $\deg(W)$ is even for $W \neq V$ and $V'$.

The counting at $V$ is the same except for the starting edge along which $\alpha$ leaves $V$. If $V' \neq V$, then $\alpha$ does not end at $V$, and this extra edge gives us

$$\deg(V) = 1 + 2(\text{number of times } \alpha \text{ comes back to } V \text{ after starting from } V).$$

Similarly, $V' \neq V$ implies

$$\deg(V') = 2(\text{number of times } \alpha \text{ reaches } V' \text{ before finally ending at } V') + 1.$$

In particular, if $V' \neq V$, then the degrees at $V$ and $V'$ are odd.

If $V = V'$, then

$$\deg(V) = 2(\text{number of times } \alpha \text{ passes } V) + 2,$$

where the last 2 counts the edges along which $\alpha$ starts from and ends at $V$. Therefore all vertices have even degrees. □

The proof of the other direction relies on the following technical result.

**Lemma 3.2.1.** *If all vertices of a connected graph $G$ have even degrees. Then deleting any one edge from $G$ still produces a connected graph.*

*Proof.* Suppose $G'$ is obtained by deleting one edge $E$ from $G$. If $G'$ were not connected, then $G'$ would have two connected components $G_1$, $G_2$, and $E$ connects a vertex $V \in G_1$ to a vertex $V' \in G_2$. Moreover, for vertices $W \in G_1$, we have

$$\deg_{G_1}(W) = \begin{cases} \deg_G(W), & \text{if } W \neq V, \\ \deg_G(V) - 1, & \text{if } W = V. \end{cases}$$

In particular, $G_1$ contains exactly one vertex of odd order degree. This implies that the sum of degrees over all vertices of $G_1$ would be odd, a contradiction to Proposition 3.1.3. The contradiction implies that $G'$ must be connected. □

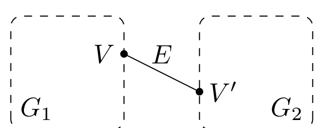



**Figure 3.2.2.** *What happens when deleting $E$ produces a non-connected graph?*

Now we return to the proof of Theorem 3.1.2.

*Proof of* $2 \implies 1$. Inspired by the proof of the $1 \implies 2$ part, we will actually try to prove the following more detailed claims for a connected graph:

1. If all degrees are even, then starting from any vertex, there is a cycle in which each edge appears exactly once.
2. If there are exactly two vertices of odd degrees, then starting from any vertex of odd degree, there is a path in which each edge appears exactly once (in fact the path must end at the other vertex of odd degree).

The claims obviously hold for graphs with only one edge. Now assume $n \geq 2$ and the claims hold for all connected graphs with number of edges $< n$. Consider a connected graph $G$ with $n$ edges.

Pick an edge $E$ and denote by $V$, $V'$ the end vertices of the edge. In case $G$ has vertices of odd degrees, we additionally assume that $\deg_G(V)$ is odd. By removing $E$ from $G$, we get a graph $G'$ with $n-1$ edges.

To apply the inductive assumption to $G'$, we need to make sure that $G'$ is connected. If $E$ is a loop, then $G'$ is clearly connected. By Lemma 3.2.1, if all vertices of $G$ have even degrees, then $G'$ is also connected. It remains to consider the case $G$ has exactly two vertices of odd degrees, and $E$ is not a loop. If $G'$ were not connected, then $G'$ would have two connected components $G_1$, $G_2$, with $V \in G_1$, as depicted in Figure 3.2.2. We find $\deg_{G_1}(V) = \deg_G(V) - 1$ to be even, and $\deg_{G_1}(W) = \deg_G(W)$ for the vertices $W \in G_1$ that are different from $V$. Since $G$ has exactly two vertices of odd degrees, and $V$ is already one of the two, we see that $G_1$ has at most one vertex of odd degree. As pointed out in the proof of Lemma 3.2.1, a graph cannot have just one vertex of odd degree. Therefore $G_1$ has no vertex of odd degree. By Lemma 3.2.1, deleting an edge $E'$ of $G_1$ emanating from $V$ would produce a connected graph. Consequently, deleting $E'$ from $G$ will also produce a connected graph. We conclude that, by changing the choice of $E$ if necessary, we may always assume that $G'$ is connected.

It remains to find a path of $G'$ that starts from either $V$ or $V'$ and contains each edge of $G'$ exactly once. Adding $E$ to such a path produces a path in which each edge of $G$ appears exactly once, and this would complete the proof.

If $E$ is a loop, then $V = V'$ and

$$\deg_{G'}(W) = \begin{cases} \deg_G(W), & \text{if } W \neq V, \\ \deg_G(V) - 2, & \text{if } W = V. \end{cases}$$

Therefore similar to $G$, the graph $G'$ either has no vertex of odd degree or has exactly two vertices of odd degrees. Moreover, $\deg_{G'}(V)$ is odd in the latter case. Applying the inductive assumption to $G'$, there is a path in $G'$ that starts from $V$ and contains each edge of $G'$ exactly once.

If $E$ is not a loop, then $V \neq V'$ and

$$\deg_{G'}(W) = \begin{cases} \deg_G(W), & \text{if } W \neq V \text{ and } V', \\ \deg_G(V) - 1, & \text{if } W = V \text{ or } V'. \end{cases}$$

Thus we have the following possibilities for $G'$:

1. If all vertices of $G$ have even degrees, then $V$ and $V'$ are the only two vertices of $G'$ of odd degrees.

2. If $V$ and $V'$ are the only two vertices of $G$ of odd degrees, then all vertices of $G'$ have even degrees.

3. If $V$ and $V'' \neq V'$ are the only two vertices of $G$ of odd degrees, then $V'$ and $V''$ are the only two vertices of $G'$ of odd degrees.

In each case, applying the inductive assumption to $G'$ produces a path of $G'$ that starts from $V'$ and contains each edge of $G'$ exactly once. □

Exercise 3.2.1. Which graph in Figure 3.2.3 admits a path in which each edge appears exactly once? If possible, find such a path.

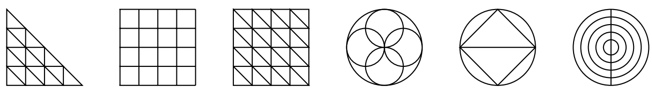


**Figure 3.2.3.** *Can you make one trip?*

Exercise 3.2.2. Construct one more bridge in Königsberg, so that Euler's problem can be solved.

Exercise 3.2.3. Given a connected graph, how many edges do you need to add in order for Euler's problem to have solutions?

Exercise 3.2.4. What if you are allowed to take two trips instead of just one?

## 3.3 Euler Formula

The next step in freeing mathematics from being a subject about measurement was also due to Euler. In a 1750 letter he wrote to Goldbach[5], Euler gave his famous formula for a connected graph on a 2-dimensional sphere

$$v - e + f = 2,$$

where $v$ is the number of vertices, $e$ is the number of edges and $f$ is the number of faces (regions of the sphere divided by the graph).

It is interesting to realize that this, really rather simple, formula seems to have been missed by Archimedes and Descartes although both wrote extensively on polyhedra. Again the reason must be that to everyone before Euler, it had been impossible to think of geometrical properties without measurement being involved.

By punching a hole on the sphere, we get a plane. As a result, any graph on the sphere and not touching the hole becomes a graph on the plane. This *planar*

[5]Christian Goldbach, born March 18, 1690 in Königsberg, Prussia (now Kaliningrad, Russia), died November 20, 1764 in Moscow, Russia. Goldbach is best remembered for his conjecture, made in 1742 in a letter to Euler, that every even integer greater than 2 is the sum of two primes. The conjecture is still open.

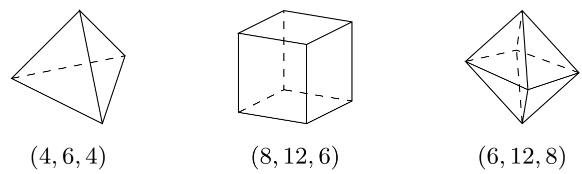



**Figure 3.3.1.** *counting vertices, edges, and faces:* $(v, e, f)$

graph encloses several finite regions and leaves one infinite outside region. The number $f$ of finite regions for a planar graph is one less than the corresponding number for the graph on the sphere. Therefore the Euler formula for planar graphs is the following.

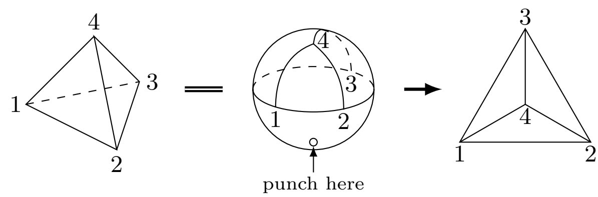



**Figure 3.3.2.** *converting spherical Euler formula to planar Euler formula*

**Theorem 3.3.1.** *For any connected planar graph,* $v - e + f = 1$.

*Proof.* The proof is by induction on the number $e$ of edges.

There are two possible graphs with $e = 1$: (i) The edge has two distinct end vertices, for which $v = 2$, $e = 1$, $f = 0$; (ii) The edge is a loop, for which $v = 1$, $e = 1$, $f = 1$. The Euler formula holds in both cases.

Assume the formula holds for graphs with $e < n$. Let $G$ be a connected graph on the plane with $e = n$.

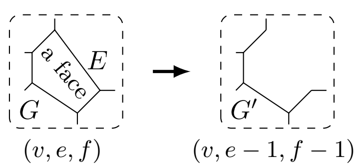



**Figure 3.3.3.** *proof of Euler formula in case* $f \neq 0$

If $f \neq 0$, then there are finite regions. Let $E$ be an edge on the boundary of one finite region. Removing $E$ produces a new graph $G'$ that is still connected. Since $E$ is on the boundary of a finite region, $G'$ has one less finite region than $G$. Of course $G'$ also has one less edge and the same number of vertices. Therefore $f(G') = f(G) - 1$, $e(G') = e(G) - 1$, $v(G') = v(G)$. Since $e(G') = n - 1 < n$, the inductive assumption implies $v(G') - e(G') + f(G') = 1$. This further implies $v(G) - e(G) + f(G) = 1$.

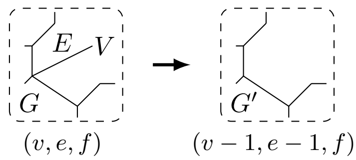



**Figure 3.3.4.** *proof of Euler formula in case* $f = 0$

If $f = 0$, then there is no finite region. We claim that there must be a vertex with degree 1. If not, then all the vertices have degrees $\geq 2$. Because any vertex has at least two edges emanating from it, a path can be constructed without "backtracking": going from one end of an edge to the other end and then coming back. Because the number of vertices is finite, sooner or later the path must cross itself at some vertex. In this way, a cycle is found to enclose a finite region. Although the finite region may be further divided by the graph, we must have $f > 0$ anyway.

Now let $V$ be a vertex of degree one. Then only one edge $E$ emanates from it. Removing $V$ and $E$ produces a new graph $G'$ which is still connected. Moreover, $f(G') = f(G)$, $e(G') = e(G) - 1$, $v(G') = v(G) - 1$. Since $e(G') = n - 1 < n$, the inductive assumption implies $v(G') - e(G') + f(G') = 1$. This further implies $v(G) - e(G) + f(G) = 1$. □

Exercise 3.3.1. Prove that for any connected graph, there is an edge so that deleting the edge still produces a connected graph. In case the graph has vertices of odd degrees, can you choose the deleted edge so that one end has odd degree?

## 3.4 Application of Euler Formula

This section contains two applications of the Euler formula.

Consider three houses and three wells in Figure 3.4.1. Can you construct one path between each house and each well, such that any two paths do not cross each other? Think of the houses and wells as vertices and the paths connecting them as edges. Then the situation may be abstracted to the graph $K_{3,3}$ on the left of Figure 3.4.2.

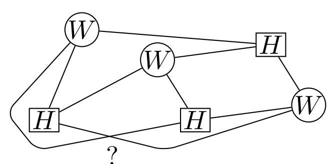



**Figure 3.4.1.** *three houses and three wells*

The answer to the question is no, according to the following famous theorem.

**Theorem 3.4.1** (Kuratowski[6] Theorem). *A graph is planar if and only if it does not contain any of the graphs $K_{3,3}$ and $K_5$ in Figure 3.4.2.*

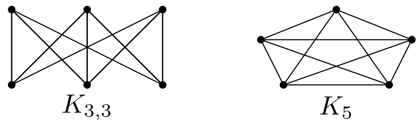



**Figure 3.4.2.** *basic non-planar graphs*

The proof of the theorem is rather complicated. Only the proof of the necessary part is given here.

*Proof of Necessary Part.* Since any part of a planar graph is also planar, the necessary part simply means that $K_5$ and $K_{3,3}$ are not planar. Moreover, by embedding the plane into a sphere, a planar graph can also be embedded into a sphere.

Suppose $K_5$ is planar. Then $K_5$ may be embedded into a sphere, so that the sphere is divided into several faces. By the Euler formula $v - e + f = 2$ for the sphere and $e = 10$, $v = 5$, we have $f = 7$. Since the boundary of any face is a cycle, which in $K_5$ must consist of at least three edges, the boundary of each face consists of at least three edges. On the other hand, each edge is shared by exactly two faces. The consideration leads to the inequality $2e \geq 3f$, which is not satisfied by $e = 10$ and $f = 7$. The contradiction shows that $K_5$ is not planar.

The proof for $K_{3,3}$ not to be planar is similar. By embedding the graph into the sphere, we have $e = 9$, $v = 6$. Then from $v - e + f = 2$, we also have $f = 5$. Again the boundary of any face is a cycle, which in $K_{3,3}$ must consist of at least four edges. Therefore we should have $2e \geq 4f$, which contradicts with $e = 9$, $f = 5$. The contradiction shows that $K_{3,3}$ is also not planar. $\square$

Exercise 3.4.1. Construct embeddings of $K_{3,3}$ and $K_5$ in the torus.

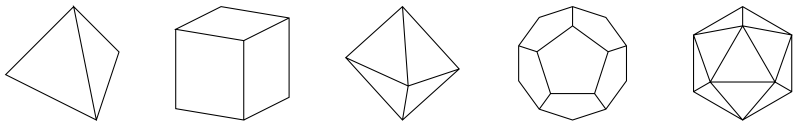
Exercise 3.4.2. Let $G$ be a graph embedded in a sphere, dividing the sphere into many faces. For each face $F$, let the degree $\deg(F)$ of the face to be the number of edges on the boundary of $F$. Prove that $\sum_{\text{all faces } F} \deg(F) = 2e$.

Now we turn to the other application.

The five Platonic[7] solids are given in Figure 3.4.3. They are characterized by the extreme symmetry in numbers: All the faces have the same number of edges on the boundary. All the vertices have the same number of edges emanating from them (i.e., same degree).

[6]Kazimierz Kuratowski, born February 2, 1896 in Warsaw, Poland, died June 18, 1980 in Warsaw, Poland. Kuratowski made many important contributions to topology and set theory.

[7]Plato, born 427 BC in Athens, Greece, died 347 BC in Athens, Greece. Plato founded, on land which had belonged to Academos, a school of learning called the Academy. The institution was devoted to research and instruction in philosophy and sciences. Over the door of the Academy was written: *Let no one unversed in geometry enter here.*



tetrahedron cube octohedron dodecahedron icosahedron

**Figure 3.4.3.** *Platonic solids*

**Theorem 3.4.2.** *The five Platonic solids are the only ones with such numerical symmetry.*

*Proof.* We need to show the table lists the only possible values for $v, e, f$.

| | $v$ | $e$ | $f$ |
|---|---|---|---|
| tetrahedron | 4 | 6 | 4 |
| cube | 8 | 12 | 6 |
| octahedron | 6 | 12 | 8 |
| icosahedron | 20 | 30 | 12 |
| dodecahedron | 12 | 30 | 20 |

Suppose each face has $m$ edges on the boundary, and each vertex has $n$ edges emanating from it. The geometric consideration tells us that we should assume $m$, $n$ and $e$ to be $\geq 3$. Then by Proposition 3.1.3,

$$nv = 2e.$$

Moreover, if the number of edges is counted from the viewpoint of faces, we get $mf$. However, because each edge is on the boundary of exactly two faces, this counts each edge twice. Therefore

$$mf = 2e.$$

Substituting these into Euler formula, we get

$$v - e + f = \frac{2e}{n} - e + \frac{2e}{m} = 2,$$

or

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{m} = \frac{1}{e}.$$

The equation will be solved under the assumption that $m$, $n$, $e$ are integers $\geq 3$.

The equation implies

$$\frac{2}{\min\{m, n\}} \geq \frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{e} > \frac{1}{2}.$$

Therefore $\min\{m, n\} < 4$. Combined with the assumption $m, n \geq 3$, we get $\min\{m, n\} = 3$.

Suppose $m = 3$. Then the equation becomes

$$\frac{1}{n} - \frac{1}{6} = \frac{1}{e} > 0.$$

Thus $n < 6$, and we get the following possibilities:

1. $m = 3, n = 3$. Then $e = 6$ and $v = 4, f = 4$.
2. $m = 3, n = 4$. Then $e = 12$ and $v = 6, f = 8$.
3. $m = 3, n = 5$. Then $e = 30$ and $v = 12, f = 20$.

If $n = 3$ instead, then $m$ and $n$ may be exchanged in the discussion above. As a result, $v = \dfrac{2e}{n}$ and $f = \dfrac{2e}{m}$ are also exchanged, and the following are all the possibilities:

1. $m = 3, n = 3$. Then $e = 6$ and $v = 4, f = 4$.
2. $m = 4, n = 3$. Then $e = 12$ and $v = 8, f = 6$.
3. $m = 5, n = 3$. Then $e = 30$ and $v = 20, f = 12$.

Combining the six cases together, we get the five cases in the table. □

Exercise 3.4.3. What is the condition on positive integers $v$, $e$, $f$ that can be realized by a connected graph embeded in the sphere?

Exercise 3.4.4. Suppose a connected graph is embedded in a sphere such that any face has exactly three edges. Prove that $e = 3v - 6$, $f = 2v - 4$.

Chapter 4

# Topology

## 4.1 Topological Basis and Subbasis

The key topological concepts for metric spaces can be introduced from balls. In fact, a careful examination of the definitions and theorems for metric spaces indicates that exactly two key properties about the balls were used. Therefore even without metric, a similar topological theory can be developed as long as a system of balls satisfying these two properties is given. This observation leads to the concept of topological basis.

**Definition 4.1.1.** A *topological basis* on a set $X$ is a collection $\mathcal{B}$ of subsets of $X$, such that

- If $x \in X$, then $x \in B$ for some $B \in \mathcal{B}$.
- If $x \in B_1 \cap B_2$ and $B_1, B_2 \in \mathcal{B}$, then $x \in B \subset B_1 \cap B_2$ for some $B \in \mathcal{B}$.

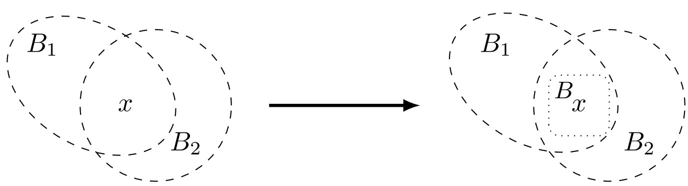



**Figure 4.1.1.** *definition of topological basis*

The subsets in $\mathcal{B}$ may be considered as "generalized balls" and do not have to be produced from a metric. The following lemma is a useful and simple way of recognizing topological bases. (The converse is however not true.) The lemma can be proved by taking $B = B_1 \cap B_2$ in the second condition for topological basis.

**Lemma 4.1.2.** *Suppose a collection $\mathcal{B}$ satisfies the first condition and*

$$B_1, B_2 \in \mathcal{B} \implies B_1 \cap B_2 \in \mathcal{B} \text{ or } B_1 \cap B_2 = \emptyset.$$

*Then $\mathcal{B}$ is a topological basis.*

**Example 4.1.1.** On any set $X$, the collections of all subsets, all finite subsets, and all single point subsets are topological bases. The collection $\{\emptyset, X\}$ is also a topological basis.

**Example 4.1.2.** The collection $\mathcal{B} = \big\{\{1\}, \{4\}, \{1,2\}, \{1,3\}\big\}$ of subsets of $X = \{1,2,3,4\}$ is a topological basis by Lemma 4.1.2. However, the collection $\mathcal{S} = \big\{\{4\}, \{1,2\}, \{1,3\}\big\}$ is not a topological basis. It violates the second condition with $x = 1$, $B_1 = \{1,2\}$, $B_2 = \{1,3\}$.

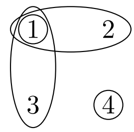



**Figure 4.1.2.** *a topological basis on four points*

**Example 4.1.3.** By Lemma 4.1.2, the following collections are all topological bases on $\mathbb{R}$.

$$\mathcal{B}_1 = \{(a,b)\colon a<b\}, \qquad \mathcal{B}_2 = \{[a,b)\colon a<b\},$$
$$\mathcal{B}_3 = \{(a,b]\colon a<b\}, \qquad \mathcal{B}_4 = \mathcal{B}_1 \cup \mathcal{B}_2,$$
$$\mathcal{B}_5 = \{(a,\infty)\colon \text{all } a\}, \qquad \mathcal{B}_6 = \{(-\infty,a)\colon \text{all } a\}$$
$$\mathcal{B}_7 = \{(a,b)\colon a<b \text{ and } a,b\in\mathbb{Q}\}, \qquad \mathcal{B}_8 = \{[a,b)\colon a<b \text{ and } a,b\in\mathbb{Q}\},$$
$$\mathcal{B}_9 = \{\mathbb{R}-F\colon F \text{ is finite}\}, \qquad \mathcal{B}_{10} = \{[a,b]\colon a\le b\}.$$

On the other hand, $\mathcal{B}_2\cup\mathcal{B}_3$, $\mathcal{S}=\{\text{open intervals of length } 1\}$ and $\mathcal{S}'=\{[a,b]\colon a<b\}$ are not topological bases. To see $\mathcal{B}_2\cup\mathcal{B}_3$ is not a topological basis, consider $0\in(-1,0]\cap[0,1)$. The only subset $B$ satisfying $0\in B\subset(-1,0]\cap[0,1)$ is $\{0\}$, which is not in $\mathcal{B}_2\cup\mathcal{B}_3$. Therefore the second condition for topological basis is not satisfied.

**Example 4.1.4.** The following collections are all topological bases on $\mathbb{R}^2$.

$$\begin{aligned}
\mathcal{B}_1 &= \{(a,b)\times(c,d)\colon a<b, c<d\},\\
\mathcal{B}_2 &= \{\text{open disks}\},\\
\mathcal{B}_3 &= \{\text{open triangles}\},\\
\mathcal{B}_4 &= \{[a,b)\times[c,d)\colon a<b, c<d\},\\
\mathcal{B}_5 &= \{[a,b)\times(c,d)\colon a<b, c<d\},\\
\mathcal{B}_6 &= \{(a,b)\times(c,d)\colon a<b, c<d, \text{ and } a,b,c,d\in\mathbb{Q}\}.
\end{aligned}$$

Lemma 4.1.2 can be used to show that $\mathcal{B}_1$, $\mathcal{B}_4$, $\mathcal{B}_5$, $\mathcal{B}_6$ are topological bases. Although the lemma cannot be applied to $\mathcal{B}_2$ and $\mathcal{B}_3$, they are still topological bases (see Figure 4.1.3).

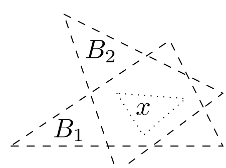



**Figure 4.1.3.** $\mathcal{B}_3$ *is a topological basis.*

**Example 4.1.5.** For any integer $n>0$, numbers $a_1,\dots,a_n$, points $t_1,\dots,t_n\in[0,1]$, and $\epsilon>0$, define

$$B(a_1,\dots,a_n,t_1,\dots,t_n,\epsilon)=\{f\in C[0,1]\colon |f(t_1)-a_1|<\epsilon,\dots,|f(t_n)-a_n|<\epsilon\}.$$

The collection $\mathcal{B} = \{B(a_1, \dots, a_n, t_1, \dots, t_n, \epsilon)\colon$ all $n, a_1, \dots, a_n, t_1, \dots, t_n, \epsilon\}$ is a topological basis on $C[0, 1]$.

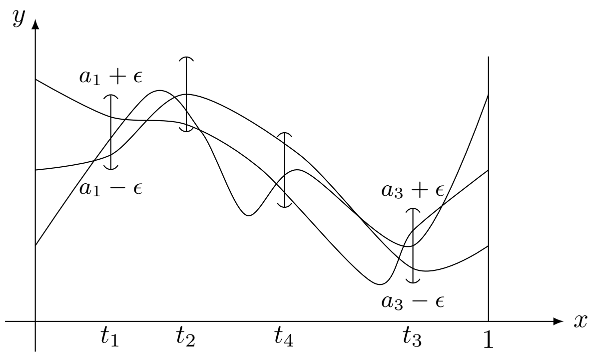



**Figure 4.1.4.** $B(a_1, \dots, a_n, t_1, \dots, t_n, \epsilon)$

Exercise 4.1.1. For any set $X$, prove that $\mathcal{B}_1 = \{X - F\colon F \subset X \text{ is finite}\}$ and $\mathcal{B}_2 = \{X - C\colon C \subset X \text{ is countable}\}$ are topological bases.

Exercise 4.1.2. Prove the balls in a metric space form a topological basis. More precisely, prove that if $d(x, x_1) < \epsilon_1$ and $d(x, x_2) < \epsilon_2$, then there is $\epsilon > 0$, such that $d(x, y) < \epsilon$ implies $d(y, x_1) < \epsilon_1$ and $d(y, x_2) < \epsilon_2$.

Exercise 4.1.3. Suppose $\mathcal{B}$ and $\mathcal{B}'$ are topological bases on $X$. Which are also topological bases?

1. $\mathcal{B} \cup \mathcal{B}' = \{B\colon B \in \mathcal{B} \text{ or } \mathcal{B}'\}$.
2. $\mathcal{B} \cap \mathcal{B}' = \{B\colon B \in \mathcal{B} \text{ and } \mathcal{B}'\}$.
3. $\mathcal{B} - \mathcal{B}' = \{B\colon B \in \mathcal{B} \text{ and } B \notin \mathcal{B}'\}$.
4. $\{B \cup B'\colon B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$.
5. $\{B \cap B'\colon B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$.
6. $\{B - B'\colon B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$.

Exercise 4.1.4. Suppose $f\colon X \to Y$ is a map and $\mathcal{B}$ is a topological basis on $Y$. Is $f^{-1}(\mathcal{B}) = \{f^{-1}(B)\colon B \in \mathcal{B}\}$ a topological basis on $X$? Is $\{B\colon f(B) \in \mathcal{B}\}$ a topological basis on $X$?

Exercise 4.1.5. Suppose $f\colon X \to Y$ is a map and $\mathcal{B}$ is a topological basis on $X$. Is $f(\mathcal{B}) = \{f(B)\colon B \in \mathcal{B}\}$ a topological basis on $Y$? Is $\{B\colon f^{-1}(B) \in \mathcal{B}\}$ a topological basis on $Y$?

Exercise 4.1.6. Suppose $\mathcal{B}_X$ and $\mathcal{B}_Y$ are topological bases on $X$ and $Y$. Is $\mathcal{B}_X \times \mathcal{B}_Y = \{B_1 \times B_2\colon B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ a topological basis on $X \times Y$?

Given any collection $\mathcal{S}$ of subsets of $X$, the collection

$$\mathcal{B} = \{S_1 \cap \dots \cap S_n\colon S_i \in \mathcal{S}, n \geq 1\} \cup \{X\}$$

is, by Lemma 4.1.2, a topological basis. The collection $\mathcal{S}$ is called the *topological* *subbasis* that induces (or generates) $\mathcal{B}$. Note that $X$ is added to the collection for the sole purpose of making sure that the first condition for topological basis is satisfied. Therefore there is no need to add $X$ in case $\cup_{S\in\mathcal{S}}S = X$.

**Example 4.1.6.** On any set $X$, the subbasis $\mathcal{S} = \{X - \{x\}\colon \text{ all } x \in X\}$ induces the topological basis $\mathcal{B} = \{X - F\colon F \subset X \text{ is finite}\}$ in Exercise 4.1.1.

**Example 4.1.7.** Consider the collection $\mathcal{S} = \{\text{open intervals of length } 1\}$ as a topological subbasis on $\mathbb{R}$. The topological basis induced by $\mathcal{S}$ is $\mathcal{B} = \{\text{open intervals of length } \le 1\}$. Note that there is no need to include the whole space $\mathbb{R}$ here.

Alternatively, take $\mathcal{S} = \mathcal{B}_5 \cup \mathcal{B}_6$ in Example 4.1.3. Since $(a, b) = (-\infty, b) \cap (a, \infty)$, $\mathcal{S}$ induces the topological basis $\mathcal{B}_1$ in the example.

**Example 4.1.8.** In Example 4.1.5, a topological basis on $C[0, 1]$ is constructed by considering arbitrary number of points in $[0, 1]$. If only one point is taken, then the corresponding collection

$$\mathcal{S} = \{B(a, t, \epsilon)\colon \text{ all } a, t, \epsilon\}$$

is no longer a topological basis. The topological basis induced by $\mathcal{S}$ consists of subsets of the form

$$B(a_1, t_1, \epsilon_1) \cap \cdots \cap B(a_n, t_n, \epsilon_n),$$

where $n$ is arbitrary. The topological basis in Example 4.1.5 consists of such subsets with $\epsilon_1 = \cdots = \epsilon_n$. However, it will be shown in Example 4.4.8 that the two topological bases produce the same topological theory on $C[0, 1]$.

Exercise 4.1.7. Find topological subbases that induce the topological bases $\mathcal{B}_2$ and $\mathcal{B}_3$ in Example 4.1.3.

Exercise 4.1.8. Prove that the subbasis $\mathcal{S} = \{(a, b) \times \mathbb{R}\colon a < b\} \cup \{\mathbb{R} \times (c, d)\colon c < d\}$ induces the topological basis $\mathcal{B}_1$ in Example 4.1.4. Find similar topological subbases for $\mathcal{B}_4$, $\mathcal{B}_5$ and $\mathcal{B}_6$ in the example.

Exercise 4.1.9. For any $n \in \mathbb{N}$, let $n\mathbb{N} = \{nk\colon k \in \mathbb{N}\}$ be the set of all multiples of $n$. For example, $2\mathbb{N}$ is the set of all even numbers. Describe the topological basis on $\mathbb{N}$ induced by the subbasis $\mathcal{S} = \{p\mathbb{N}\colon p \text{ is a prime number}\}$.

## 4.2 Open Subset

The concept of open subsets may be extended by using topological basis in place of balls.

**Definition 4.2.1.** A subset $U \subset X$ is *open* with respect to a topological basis $\mathcal{B}$ if it satifies

$$x \in U \implies x \in B \subset U \text{ for some } B \in \mathcal{B}.$$

**Example 4.2.1.** On any set $X$, any subset is open with respect to the topological bases

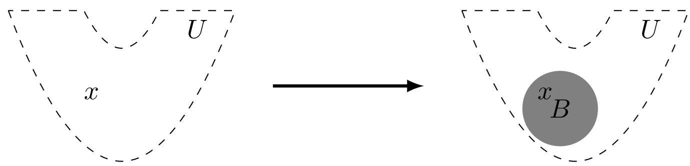



**Figure 4.2.1.** *definition of open subset*

of all subsets, all finite subsets, and all single point subsets. On the other hand, the only subsets that are open with respect to the topological basis $\{\emptyset, X\}$ are $\emptyset$ and $X$.

**Example 4.2.2.** Consider the topological basis in Example 4.1.2. Since

$$1 \in \{1\} \subset \{1,2,3\}, \quad 2 \in \{1,2\} \subset \{1,2,3\}, \quad 3 \in \{1,3\} \subset \{1,2,3\},$$

the subset $\{1,2,3\}$ is open. Moreover, there is no subset in the topological basis lying between 2 and $\{2,3\}$, so that $\{2,3\}$ is not open.

**Example 4.2.3.** If $U \in \mathcal{B}$, then we may take $B = U$ in Definition 4.2.1. This shows that any subset in a topological basis is open with respect to the topological basis. The fact extends Lemma 2.3.2 for metric spaces.

**Example 4.2.4.** We try to determine whether the interval $(0,1)$ is open with respect to some of the topological bases in Example 4.1.3.

For any $x \in (0,1)$, we have $x \in B \subset (0,1)$ by choosing $B = (0,1) \in \mathcal{B}_1$ (see Example 4.2.3). Therefore $(0,1)$ is open in $\mathcal{B}_1$. For any $x \in (0,1)$, we have $x \in B \subset (0,1)$ by choosing $B = [x,1) \in \mathcal{B}_2$. Therefore $(0,1)$ is open in $\mathcal{B}_2$. There is no $B \in \mathcal{B}_5$ satisfying $x \in B \subset (0,1)$, because any $B \in \mathcal{B}_5$ is an unbounded interval. Therefore $(0,1)$ is not open in $\mathcal{B}_5$.

**Example 4.2.5.** On the left of Figure 4.2.2 is an open triangle plus one open (meaning excluding end points) edge. This is open with respect to the topological basis $\mathcal{B}_4$ in Example 4.1.4. For the triangle in the middle, the location of the edge is changed and the subset is no longer open. On the right, the edge is changed from open to closed (meaning including end points). We find that the subset is also not open with respect to $\mathcal{B}_4$.

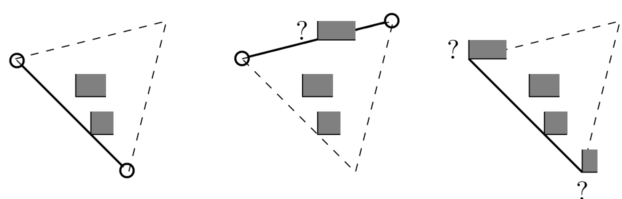



**Figure 4.2.2.** *open subset in $\mathcal{B}_4$*

**Example 4.2.6.** In Example 2.3.4, we know the subset $U$ of all positive functions is $L_\infty$-open but not $L_1$-open. Now we show that $U$ is not open with respect to the topological basis in Example 4.1.5. This means that there is $f \in U$, such that $f \in B = B(a_1, \dots, a_n, t_1, \dots, t_n, \epsilon)$ implies $B \not\subset U$. In fact, we will show that any such $B$ contains functions that are negative somewhere. Therefore we always have $B \not\subset U$, regardless of $f$.

Given $B$, we choose any $f \in B$ and a point $t_0$ distinct from any of $t_1, \dots, t_n$. Then $|f(t_1) - a_1| < \epsilon$, $\dots$, $|f(t_n) - a_n| < \epsilon$. By modifying $f$ around $t_0$, it is not difficult to construct a continuous function $g$ satisfying

$$g(t_0) = -1;\ g(t_1) = f(t_1),\ \dots,\ g(t_n) = f(t_n).$$

Then $g \in B$ and $g \notin U$.

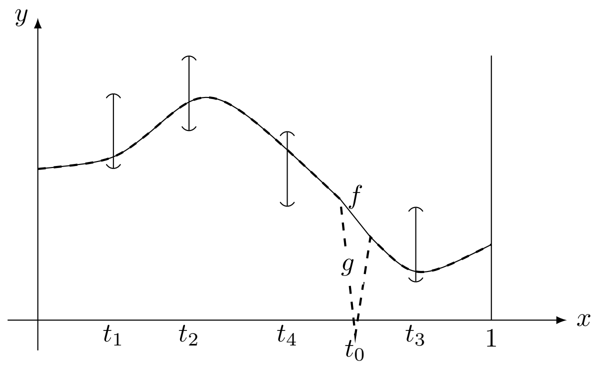



**Figure 4.2.3.** *Positive functions do not form pointwise convergent open subset.*

In Example 2.4.8, we know the subset $V = E^{-1}(1, \infty) = \{f \in C[0,1] \colon f(0) > 1\}$ is also $L_\infty$-open but not $L_1$-open. Yet the subset is open with respect to the topological basis in Example 4.1.5 because $f \in V$ implies $\epsilon = f(0) - 1 > 0$, and then we have $f \in B(f(0), 0, \epsilon) \subset V$.

Exercise 4.2.1. Determine open subsets.

1. $\{2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$, with respect to the topological basis in Example 4.1.2.
2. $[0,1]$, $(0,1]$, $(0,+\infty)$, with respect to the topological bases in Example 4.1.3.
3. $\{f \colon f(0) > f(1)\}$, $\left\{f \colon \int_0^1 f(t)dt < 1\right\}$, $\left\{f \colon \int_0^{\frac{1}{2}} f(t)dt > \int_{\frac{1}{2}}^1 f(t)dt\right\}$, with respect to the topological basis in Example 4.1.5.
4. Disks in Figure 4.2.4 plus part of the open boundary arc, with respect to the topological bases in Example 4.1.4.

Exercise 4.2.2. Prove that a subset $U \subset \mathbb{N}$ is open with respect to the topological subbasis in Exercise 4.1.9 if and only if $p_1^{n_1} \cdots p_k^{n_k} \in U$ for distinct primes $p_i$ and natural numbers $n_i$ implies that $mp_1 \cdots p_k \in U$ for any natural number $m$. In particular, $n\mathbb{N}$ is open if and only if $n$ contains no square factors.

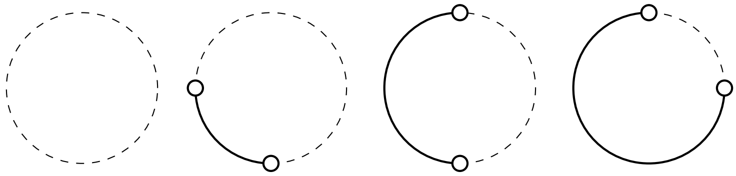


**Figure 4.2.4.** *Which are open?*

Exercise 4.2.3. Prove that $\mathcal{B} = \{n\mathbb{N} : n \in \mathbb{N}\}$ is a topological basis on $\mathbb{N}$. Then describe open subsets with respect to the topological basis.

Exercise 4.2.4. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be topological bases on $X$ and $Y$. Show that $\mathcal{B}_X \cup \mathcal{B}_Y$ is a topological basis on the disjoint union $X \sqcup Y$. What are the open subsets of $X \sqcup Y$ with respect to this topological basis?

Exercise 4.2.5. Prove that Definitions 2.3.1 and 4.2.1 give the same open subsets for metric spaces. The subtle point here is that $x$ is not necessarily the center of ball in Definition 4.2.1.

Proposition 2.3.3 lists properties of open subsets in metric spaces. All the properties can be extended.

**Proposition 4.2.2.** *The open subsets with respect to a topological basis on $X$ satisfy the following:*

1. *$\emptyset$ and $X$ are open.*
2. *Unions of open subsets are open.*
3. *Finite intersections of open subsets are open.*

*Proof.* By default, $\emptyset$ is open because there is no point in $\emptyset$ for us to verify the condition. On the other hand, the first condition for topological basis implies $X$ is open.

The second property can be proved in the same way as in Proposition 2.3.3.

The proof of the third property is also similar. Let $U_1$, $U_2$ be open, and let $x \in U_1 \cap U_2$. Then $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$ for some $B_1, B_2 \in \mathcal{B}$. By the second condition for topological basis, we have $x \in B \subset B_1 \cap B_2 \subset U_1 \cap U_2$ for some $B \in \mathcal{B}$. $\square$

The following extends Lemma 2.3.4 and gives another description of open subsets with respect to a topological basis.

**Lemma 4.2.3.** *A subset is open with respect to a topological basis if and only if it is a union of subsets in the topological basis.*

*Proof.* By Example 4.2.3 and the second property in Proposition 4.2.2, any union of subsets in a topological basis $\mathcal{B}$ is open with respect to $\mathcal{B}$. Conversely, suppose $U$ is open with respect to $\mathcal{B}$. Then for each $x \in U$, there is $B_x \in \mathcal{B}$ such that $x \in U \in B_x$. This gives rise to

$$U = \cup_{x\in U}\{x\} \subset \cup_{x\in U} B_x \subset U,$$

which implies $U = \cup_{x\in U} B_x$, a union of subsets in $\mathcal{B}$. □

## 4.3 Topological Space

The properties of open subsets in Proposition 4.2.2 lead to the concept of topology.

**Definition 4.3.1.** A *topology* on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$, such that

- $\emptyset, X \in \mathcal{T}$.
- $U_i \in \mathcal{T} \implies \cup U_i \in \mathcal{T}$.
- $U_1, U_2 \in \mathcal{T} \implies U_1 \cap U_2 \in \mathcal{T}$.

The subsets in the collection $\mathcal{T}$ are called *open subsets*. The definition first appeared in the 1914 book "Grundzüge der Mengenlehre" by Hausdorff. In fact, an additional condition (see Definition 7.1.1) was added in the book.

A *neighborhood* of a point $x$ in a topological space is a subset $N$, such that there is an open subset $U$ satisfying $x \in U \subset N$. An *open neighborhood* of $x$ is simply an open subset containing $x$. Clearly, if $N$ is a neighborhood of $x$, then any subset bigger than $N$ is also a neighborhood of $x$. Moreover, a finite intersection of neighborhoods of $x$ is still a neighborhood of $x$.

More generally, a subset $N$ is a neighborhood of a subset $A$, if there is an open subset $U$ satisfying $A \subset U \subset N$. An open neighborhood of $A$ is simply an open subset containing $A$. The comments about neighborhoods of a point can be extended to neighborhoods of a subset.

Given a topological basis, the topology constructed by way of Definition 4.2.1 is called the topology induced from the topological basis. Lemma 4.2.3 tells us that the induced topology consists of exactly the unions of subsets in the topological basis.

**Example 4.3.1.** Any metric space is a topological space. By Exercise 2.3.9, different metrics may induce the same topology. For example, the various $L_p$-metrics on $\mathbb{R}^n$ in Example 2.1.3 are topologically the same, and the topology will be denoted by $\mathbb{R}^n_{\text{usual}}$.

**Example 4.3.2.** The topological basis $\mathcal{B}_1$ in Example 4.1.3 induces the usual topology given by the usual metric. We also denote the topology by $\mathbb{R}_{(\to)}$.

The topological basis $\mathcal{B}_2$ induces the *lower limit topology*. We denote the topology by $\mathbb{R}_{\text{lower limit}}$ or $\mathbb{R}_{[\to)}$.

The topological basis $\mathcal{B}_5$ induces the topology

$$\{(a, \infty) \colon a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

We denote the topology by $\mathbb{R}_{\leftarrow}$.

**Example 4.3.3.** On any set $X$, we have the *discrete topology* (see Example 2.3.1)

$$\mathcal{T}_{\text{discrete}} = \{\text{all subsets of } X\},$$

and the *trivial topology*

$$\mathcal{T}_{\text{trivial}} = \{\emptyset, X\}.$$

**Example 4.3.4.** The collection $\mathcal{T} = \big\{\emptyset, \{1\}\big\}$ is the only topology on the single point space $X = \{1\}$. There are four topologies on the two point space $X = \{1, 2\}$:

$$\mathcal{T}_1 = \big\{\emptyset, \{1,2\}\big\}, \qquad \mathcal{T}_2 = \big\{\emptyset, \{1\}, \{1,2\}\big\},$$
$$\mathcal{T}_3 = \big\{\emptyset, \{2\}, \{1,2\}\big\}, \qquad \mathcal{T}_4 = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}.$$

Note that if the two points are exchanged, then $\mathcal{T}_2$ and $\mathcal{T}_3$ are interchanged. Exchanging the two points is a *homeomorphism* between the two topologies.

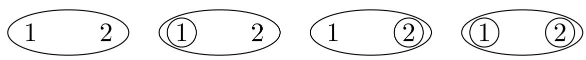



**Figure 4.3.1.** *topologies on two points*

Figure 4.3.2 contains some examples of topologies on a three point space $X = \{1, 2, 3\}$. Again some topologies may be homeomorphic. Moreover, note that some topologies are "larger" and some are "smaller". The total number of topologies on a three point space is 29. If homeomorphic topologies are counted as the same one, then the total number is 9.

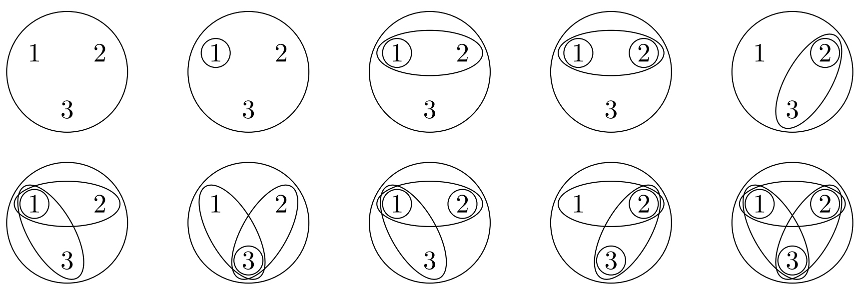


**Figure 4.3.2.** *some topologies on three points*

The exact formula for the number of topologies on a set of $n$ elements is still not known.

**Example 4.3.5.** The collection

$$\mathcal{S} = \big\{\{4\}, \{1,2\}, \{1,3\}\big\}$$

in Example 4.1.2 is not a topology on $X = \{1, 2, 3, 4\}$. To get a topology, all the finite intersections must be included. The new intersections not appearing in $\mathcal{S}$ are $\{4\} \cap \{1, 2\} =$

$\emptyset$ and $\{1,2\} \cap \{1,3\} = \{1\}$. By adding the finite intersections, $\mathcal{S}$ is enlarged to

$$\mathcal{B} = \big\{\emptyset, \{1\}, \{4\}, \{1,2\}, \{1,3\}\big\}.$$

This is basically the topological basis in Example 4.1.2, and is the topological basis induced by $\mathcal{S}$.

To get a topology, unions must also be included. In fact, according to Lemma 4.2.3, taking all possible unions of subsets in $\mathcal{B}$ already gives us the topology. The new unions not appearing in $\mathcal{B}$ are

$$\{1\} \cup \{4\} = \{1,4\}, \quad \{1,2\} \cup \{1,3\} = \{1,2,3\}, \quad \{1,2\} \cup \{4\} = \{1,2,4\},$$

$$\{1,3\} \cup \{4\} = \{1,3,4\}, \quad \{1,2\} \cup \{1,3\} \cup \{4\} = \{1,2,3,4\}.$$

By adding all of these to $\mathcal{B}$, we get a topology

$$\mathcal{T} = \big\{\emptyset, \{1\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\big\}.$$

**Example 4.3.6.** On any set $X$, we have the *finite complement topology*

$$\mathcal{T} = \{X - F \colon F \text{ is a finite subset}\} \cup \{\emptyset\}.$$

The second and the third conditions mostly follow from

$$\cup(X - F_i) = X - \cap F_i, \quad \cap_{\text{finite}}(X - F_i) = X - \cup_{\text{finite}} F_i,$$

and the fact that both $\cap F_i$ and $\cup_{\text{finite}} F_i$ are finite. I said "mostly follow" because strictly speaking, the case some $X - F_i$ are replaced by $\emptyset$ also needs to be considered.

The finite complement topology is induced by the topological subbasis in Example 4.1.6.

We make some remarks on the concept of topology.

*1. Why use open subsets to define topology?*

As seen in the theory of metric spaces (especially Propositions 2.4.2 and 2.6.2), all the key topological concepts and theories depend only on the open subsets. Therefore open subsets are more fundamental than metrics or balls, and is more suitable for the definition of the concept of topology. The situation is similar to linear algebra, where the most fundamental concepts are addition and scalar multiplication, and the properties satisfied by the two operations are used to define the concept of vector spaces.

*2. How to present a topology in practice?*

Although open subsets are used in the definition, it is almost always impractical to present specific topologies by listing all the open subsets. Instead, they are usually presented as induced from some topological basis or subbasis. Examples 4.3.3 through 4.3.6 are very rare ones in terms of the way the topologies are presented. Even for $\mathbb{R}^2_{\text{usual}}$, for example, it is really impossible to make a complete list of all the open subsets.

Because of the fundamental nature of open subsets, the definitions of subsequent topological concepts will be based on open subsets. However, for practical purpose, we always need to reinterpret the concepts in terms of topological basis or subbasis.

*3. How to construct the suitable topological (sub)basis for a specific purpose?*

In metric spaces, the balls are used to describe the closeness. As the generalization of balls, the topological basis should play the similar role. This means that the topological basis should be chosen so that a point being in one such subset matches the intuition of closeness. For example, the subbasis in Example 4.1.8 is motivated by the *pointwise convergence* of a sequence of functions $f_n$, which means the sequence $f_n(t)$ converges for any fixed $t$. Specifically, for any fixed $t$, there is a number $a$, such that for any $\epsilon > 0$, $|f_n(t) - a| < \epsilon$ holds for sufficiently large $n$. The inequality $|f_n(t) - a| < \epsilon$, for all fixed choices of $t$, $a$, $\epsilon$, gives the precise definition of the closeness we are looking for and is used in constructing $B(a, t, \epsilon)$. As a result, the subsets $B(a, t, \epsilon)$ (for all choices of $a, t, \epsilon$) should be chosen as the building blocks for the topological basis that describes the pointwise convergence. Since these subsets do not form a topological basis, the topological basis induced from these subsets (as a subbasis) is used.

Exercise 4.3.1. Which are topologies on $\mathbb{N}$? If not, are there any topological bases?

1. $\{$subsets with at least two elements$\} \cup \{\emptyset, \mathbb{N}\}$.
2. $\{$subsets with even number of elements$\} \cup \{\mathbb{N}\}$.
3. $\{$subsets containing no odd numbers$\} \cup \{\mathbb{N}\}$.
4. $\{n\mathbb{N} \colon n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$, where $n\mathbb{N}$ is defined in Exercise 4.1.9.
5. $\{2^n\mathbb{N} \colon n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$.

Exercise 4.3.2. Which are topologies on $\mathbb{R}$? If not, are there any topological bases?

1. $\{(a, \infty) \colon a \in \mathbb{Z}\} \cup \{\emptyset, \mathbb{R}\}$.
2. $\{(-\infty, -a) \cup (a, \infty) \colon a > 0\} \cup \{\emptyset, \mathbb{R}\}$.
3. $\{(-a, 2a) \colon \text{all } a > 0\} \cup \{\emptyset, \mathbb{R}\}$.
4. $\{[-a, 2a) \colon \text{all } a > 0\} \cup \{\emptyset, \mathbb{R}\}$.
5. $\{(a_n, \infty) \colon n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{R}\}$, where $a_n$ is an increasing sequence.

Exercise 4.3.3. Let $X$ be any set. Let $A \subset X$ be a subset. Prove that the following are topologies.

1. $\{X - C \colon C \text{ is a countable subset}\} \cup \{\emptyset\}$.
2. $\{U \colon U \subset A\} \cup \{X\}$.
3. $\{U \colon U \cap A = \emptyset\} \cup \{X\}$.

Exercise 4.3.4. Prove that if any single point is open, then the topology is discrete.

Exercise 4.3.5. Prove that $A$ is open if and only if for any $x \in A$, there is an open subset $U$, such that $x \in U \subset A$.

Exercise 4.3.6. Prove that $A$ is open if and only if any point in $A$ has a neighborhood contained in $A$.

Exercise 4.3.7. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on $X$. Which are also topologies?

1. $\mathcal{T} \cup \mathcal{T}' = \{U : U \in \mathcal{T} \text{ or } \mathcal{T}'\}$.
2. $\mathcal{T} \cap \mathcal{T}' = \{U : U \in \mathcal{T} \text{ and } \mathcal{T}'\}$.
3. $\mathcal{T} - \mathcal{T}' = \{U : U \in \mathcal{T} \text{ and } B \notin \mathcal{T}'\}$.
4. $\{U \cup U' : U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$.
5. $\{U \cap U' : U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$.
6. $\{U - U' : U \in \mathcal{T} \text{ and } U' \in \mathcal{T}'\}$.

Exercise 4.3.8. Suppose $f: X \to Y$ is a map and $\mathcal{T}$ is a topology on $Y$. Is $f^{-1}(\mathcal{T}) = \{f^{-1}(V) : V \in \mathcal{T}\}$ a topology on $X$? Is $\{U : f(U) \in \mathcal{T}\}$ a topology on $X$?

Exercise 4.3.9. Suppose $f: X \to Y$ is a map and $\mathcal{T}$ is a topology on $X$. Is $f(\mathcal{T}) = \{f(U) : U \in \mathcal{T}\}$ a topology on $Y$? Is $\{V : f^{-1}(V) \in \mathcal{T}\}$ a topology on $Y$?

Exercise 4.3.10. Suppose $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on $X$ and $Y$. Is $\mathcal{T}_X \times \mathcal{T}_Y = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ a topology on $X \times Y$?

Exercise 4.3.11. Let $x$ be a point in a topological space $X$. A collection $\mathcal{L}$ of open subsets is a *local topological basis* at $x$ if every $L \in \mathcal{L}$ contains $x$ and

$$x \in U \text{ for some open } U \implies x \in L \subset U \text{ for some } L \in \mathcal{L}.$$

1. Prove that if $\mathcal{B}$ is a topological basis of $X$, then $\mathcal{L} = \{B \in \mathcal{B} : x \in B\}$ is a local topological basis at $x$.
2. Prove that if a local topological basis $\mathcal{L}_x$ is given at each $x \in X$, then $\mathcal{B} = \cup_{x \in X} \mathcal{L}_x$ is a topological basis of $X$.
3. Prove that if $\mathcal{B}$ is a topological basis of $X$, then the open subset $U$ in the definition of local topological basis may be replaced by $B \in \mathcal{B}$.
4. Prove that if $X$ is a metric space, then $\mathcal{L} = \{B(x, \epsilon) : \epsilon > 0\}$ is a local topological basis at $x$. Moreover, if $\epsilon_n > 0$ is a sequence converging to 0, then $\mathcal{L} = \{B(x, \epsilon_n) : n \in \mathbb{N}\}$ is also a local topological basis at $x$.
5. Prove that $\{[x, x + \epsilon) : \epsilon > 0\}$ is a local topological basis at $x \in \mathbb{R}$ with respect to the topological basis $\mathcal{B}_2$ in Example 4.1.3.

Exercise 4.3.12. Prove that if a point has a countable local topological basis, then it has a local topological basis $\{L_n : n \in \mathbb{N}\}$ satisfying $L_{n+1} \subset L_n$. On the other hand, for uncountable $X$ with the countable complement topology (see Exercise 4.1.1), prove that no point in $X$ has countable local topological basis.

Exercise 4.3.13. Let $X$ be a topological space. A *neighborhood system* at $x$ is a collection $\mathcal{N}$ of neighborhoods of $x$, such that for any open neighborhood $U$ of $x$, there is $N \in \mathcal{N}$ satisfying $x \in N \subset U$. Thus a local topological basis is simply an open neighborhood system.

1. Prove that for any neighborhood system $\mathcal{N}$ at $x$, there is a local topological basis $\mathcal{L}$ at $x$, such that any $N \in \mathcal{N}$ contains some $L \in \mathcal{L}$.
2. Prove that any use of local topological bases (in verifying openness, for example) may be replaced by neighborhood systems.

## 4.4 Comparing Topologies

In Example 4.3.4, we saw that different topologies on the same set can sometimes be compared.

**Definition 4.4.1.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set. If $\mathcal{T} \subset \mathcal{T}'$, then we say $\mathcal{T}$ is *coarser* than $\mathcal{T}'$ and we also say $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

In other words, $\mathcal{T}$ is coarser than $\mathcal{T}'$ if

$$U \text{ is open with respect to } \mathcal{T} \implies U \text{ is open with respect to } \mathcal{T}'.$$

**Example 4.4.1.** The discrete topology is the finest topology because it is finer than any topology. The trivial topology is the coarsest topology because it is coarser than any topology.

**Example 4.4.2.** By Exercise 2.3.9, if two metrics satisfy $d'(x, y) \le c\,d(x, y)$ for a constant $c > 0$, then $d'$ induces coarser topology than $d$.

**Example 4.4.3.** In Example 4.3.5, a collection $\mathcal{S}$ of subsets not satisfying the conditions for topology is enlarged to a topology $\mathcal{T}$. The subsets added to $\mathcal{S}$ are necessary for the collection to become a topology. Since the process does not add any "unnecessary" subsets, $\mathcal{T}$ is the coarsest topology such that all subsets in $\mathcal{S}$ are open.

**Example 4.4.4.** In Example 2.3.4, we showed that the subset $U$ of all positive functions is $L_\infty$-open. In Example 4.2.6, we further showed that $U$ is not open with respect to the topological basis in Example 4.1.5. Therefore the $L_\infty$-topology is not coarser than the topology induced by the topological basis in Example 4.1.5.

The following compares topologies by using topological basis or subbasis.

**Lemma 4.4.2.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies. Suppose $\mathcal{T}$ is induced by a topological basis $\mathcal{B}$ (or a subbasis $\mathcal{S}$). Then the following are equivalent.*

1. *$\mathcal{T} \subset \mathcal{T}'$: $\mathcal{T}$ is coarser than $\mathcal{T}'$.*

2. *$\mathcal{B} \subset \mathcal{T}'$: Any $B \in \mathcal{B}$ is open with respect to $\mathcal{T}'$.*

3. *$\mathcal{S} \subset \mathcal{T}'$: Any $S \in \mathcal{S}$ is open with respect to $\mathcal{T}'$.*

The lemma may be rephrased as follows: The topology $\mathcal{T}$ induced by a topological basis $\mathcal{B}$ or subbasis $\mathcal{S}$ is the coarsest topology such that all subsets in $\mathcal{B}$ or $\mathcal{S}$ are open. In case the topology $\mathcal{T}'$ is induced by a topological basis $\mathcal{B}'$, the meaning of the second property is illustrated in Figure 4.4.1.

*Proof.* Since $\mathcal{T}$ always contains $\mathcal{B}$ or $\mathcal{S}$, the inclusion $\mathcal{T} \subset \mathcal{T}'$ implies $\mathcal{B} \subset \mathcal{T}'$ or $\mathcal{S} \subset \mathcal{T}'$.

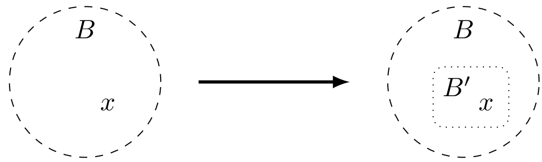



**Figure 4.4.1.** *comparing topological basis*

Conversely, assume $\mathcal{B} \subset \mathcal{T}'$. By Lemma 4.2.3, any $U \in \mathcal{T}$ is of the form $U = \cup_i B_i$, with $B_i \in \mathcal{B}$. The assumption $\mathcal{B} \subset \mathcal{T}'$ tells us that $B_i \in \mathcal{T}$. By the second condition for topology, we conclude that $U \in \mathcal{T}'$.

Finally, assume $\mathcal{S} \subset \mathcal{T}'$. The topological basis $\mathcal{B}$ induced by $\mathcal{S}$ consists of finite intersections $S_1 \cap \cdots \cap S_n$, with $S_i \in \mathcal{S}$. The assumption $\mathcal{S} \subset \mathcal{T}'$ tells us that $S_i \in \mathcal{T}$. By the third condition for topology, the intersection $S_1 \cap \cdots \cap S_n \in \mathcal{T}'$. Therefore $\mathcal{B} \subset \mathcal{T}'$. As argued above, this further implies $\mathcal{T} \subset \mathcal{T}'$. $\square$

A consequence of Lemma 4.4.2 is the following criterion for a topological basis to induce a given topology.

**Lemma 4.4.3.** *A collection $\mathcal{B}$ of subsets of a topological space $X$ is a topological basis and induces the topology on $X$ if and only if the following are satisfied.*

- *The subsets in $\mathcal{B}$ are open.*
- *For any open neighborhood $U$ of $x$, there is $B \in \mathcal{B}$, such that $x \in B \subset U$.*

In the lemma, the open neighborhood can be replaced by any neighborhood.

*Proof.* We first show that $\mathcal{B}$ is a topological basis. By taking $U = X$ in the second condition, we see that $\mathcal{B}$ satisfies the first condition for topological basis. Now suppose $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$. By the first condition, $B_1$ and $B_2$ are open in $X$. Therefore $B_1 \cap B_2$ is open in $X$. By taking $B_1 \cap B_2$ in the second condition, we get $x \in B \subset B_1 \cap B_2$ for some $B \in \mathcal{B}$. Thus the second condition for topological basis is verified.

By Lemma 4.4.2, the first condition implies that open subsets with respect to $\mathcal{B}$ are open in $X$. By the definition of openness with respect to $\mathcal{B}$, the second condition means that the open subsets in $X$ are open with respect to $\mathcal{B}$. Therefore $\mathcal{B}$ induces the topology on $X$. $\square$

**Example 4.4.5.** The coarsest topology on $\mathbb{R}$ such that $(-\infty, a)$ and $(a, \infty)$ are open for all $a$ is the usual topology.

**Example 4.4.6.** The following implication shows that the usual topology is coarser than the lower limit topology:

$$x \in (a, b) \implies x \in [x, b) \subset (a, b).$$

On the other hand, if the lower limit topology is coarser than the usual topology, then we should have

$$x \in [x,b) \implies x \in (a,c) \subset [x,b) \text{ for some } a < c.$$

However, $x \in (a,c)$ implies $a < x$ and $(a,c) \subset [x,b)$ implies $a \geq x$. The contradiction shows that the lower limit topology is not coarser than the usual topology. Therefore the usual topology is *strictly coarser* than the lower limit topology.

**Example 4.4.7.** Figure 4.4.2 shows that the topological bases $\mathcal{B}_1$ and $\mathcal{B}_3$ in Example 4.1.4 induce the same topology on $\mathbb{R}^2$. In fact, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_6$ all induce the usual topology, which we denote by $\mathbb{R}^2_{\text{usual}}$ or $\mathbb{R}^2_{\square}$.

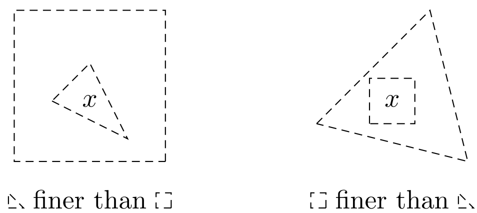



**Figure 4.4.2.** *$\mathcal{B}_1$ and $\mathcal{B}_3$ induce the same topology.*

We also denote the topologies induced by $\mathcal{B}_4$ and $\mathcal{B}_5$ as $\mathbb{R}^2_{\llcorner}$ and $\mathbb{R}^2_{\ulcorner}$. Similar to Example 4.4.6, we may show that $\mathbb{R}^2_{\square}$ is strictly coarser than $\mathbb{R}^2_{\ulcorner}$, and $\mathbb{R}^2_{\ulcorner}$ is strictly coarser than $\mathbb{R}^2_{\llcorner}$.

**Example 4.4.8.** Let $\mathcal{B}$ and $\mathcal{B}'$ be the topological bases in Examples 4.1.5 and 4.1.8.

Taking $\epsilon_1 = \cdots = \epsilon_n = \epsilon$, we have $\mathcal{B} \subset \mathcal{B}'$. Therefore $\mathcal{B}$ induces coarser topology than $\mathcal{B}'$. Conversely, $\mathcal{B}'$ is induced by the topological subbasis $\mathcal{S}$ in Example 4.1.8. Since $\mathcal{S} \subset \mathcal{B}$, $\mathcal{B}'$ induces (i.e., $\mathcal{S}$ induces) coarser topology than $\mathcal{B}$.

In conclusion, $\mathcal{B}$ and $\mathcal{B}'$ induce the same topology, called the *pointwise convergence topology* and denoted $C[0,1]_{\text{pt conv}}$.

**Example 4.4.9.** How are the $L_\infty$-topology and the pointwise convergence topology compared? By Example 2.4.8, the subsets $B(a,t,\epsilon)$ in the topological subbasis $\mathcal{S}$ in Example 4.1.8 are $L_\infty$-open. Therefore by Lemma 4.4.2, the pointwise convergence topology is coarser than the $L_\infty$-topology. Combined with Example 4.4.4, we conclude that the pointwise convergence topology is strictly coarser than the $L_\infty$-topology.

Exercise 4.4.1. Suppose $d$ is a metric on $X$. Then $\min\{d,1\}$ and $\sqrt{d}$ are also metrics (see Exercise 2.1.11). Prove that $d$, $\min\{d,1\}$ and $\sqrt{d}$ induce the same topology.

Exercise 4.4.2. Compare various topologies on $\mathbb{R}$ induced by the topological bases in Example 4.1.3. Compare the topologies on $\mathbb{N}$ given in Exercises 4.1.9, 4.3.1, and by the 2-adic metric.

Exercise 4.4.3. Use Example 2.3.4 and Exercise 2.3.11 to show that the $L_1$-topology is strictly coarser than the $L_\infty$-topology on $C[0,1]$. How can you compare the $L_p$-metrics on $C[0,1]$ for different $p$?

Exercise 4.4.4. Prove that the $L_1$-topology and the pointwise convergence topology on $C[0,1]$ cannot be compared.

1. Use Examples 2.4.8 and 4.2.6 to show that the pointwise convergence topology is not coarser than the $L_1$-topology.
2. Use the ball $B_{L_1}(0,1) = \left\{ f\colon \int_0^1 |f(t)|dt < 1 \right\}$ in Exercise 4.2.1 to show that the $L_1$-topology is not coarser than the pointwise convergence topology.

Exercise 4.4.5. Show that $\mathcal{L} = \{B(f(t_1),\dots,f(t_n),t_1,\dots,t_n,\epsilon)\colon \text{all } n,t_1,\dots,t_n,\epsilon\}$ is a local topological basis at a function $f$ in the pointwise convergence topology.

Exercise 4.4.6. Let $\mathcal{T}$ be a topology on $X$ and let $A \subset X$ be a subset. Find the coarsest topology on $X$ such that any subset in $\mathcal{T}$ is open and any single point in $A$ is open. For the case of $X = \mathbb{R}_{\leftarrow\rightarrow}$ and $A = \mathbb{Q}$, we get the *Michael line*.

Exercise 4.4.7. Prove that the subset $U$ in Lemma 4.4.3 can be replaced by subsets in a topological basis that is known to induce the topology? What if $U$ is replaced by subsets in a topological *sub*basis that is known to induce the topology?

Exercise 4.4.8. Prove that a topology $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if any neighborhood of $x$ in $\mathcal{T}$ contains a neighborhood of $x$ in $\mathcal{T}'$.

Exercise 4.4.9. Let $\mathcal{B}$ be a topological basis on $X$. Think of $\mathcal{B}$ as a topological *subbasis* and then consider the topological basis $\mathcal{B}'$ induced from $\mathcal{B}$. Prove that $\mathcal{B}$ and $\mathcal{B}'$ induce the same topology.

Exercise 4.4.10. Let $\mathcal{T}$ be a topology on $X$. Think of $\mathcal{T}$ as a topological *basis*. Prove that the topology induced by (the topological basis) $\mathcal{T}$ is still (the topology) $\mathcal{T}$.

## 4.5 Limit Point and Closed Subset

Let $A$ be a subset of a topological space. A point $x \in X$ is a *limit point* of $A$ if for any open neighborhood $U$ of $x$, there is $a$, such that

$$a \in A, \quad a \neq x, \quad a \in U.$$

The collection of limit points of $A$ is denoted $A'$.

The three conditions together means

$$(A - x) \cap U \neq \emptyset.$$

Note that if the condition is satisfied by $U$, then it is also satisfied by all subsets containing $U$. Therefore it is sufficient to verify the condition for "minimal" $U$. In particular, we only need to verify the condition only for those $U$ in a topological basis, or only for those $U$ in a neighborhood system of $x$ (see Exercise 4.3.13).

**Example 4.5.1.** We study the limit points in the topology constructed in Example 4.3.5 by making use of the topological basis $\mathcal{B}$ in Example 4.1.2.

Consider the limit points of $A = \{1\}$. The point $x = 1$ is not a limit point of $A$ because $a \in A$ and $a \neq x$ are always contradictory. The point $x = 2$ is a limit point because only $x \in B = \{1, 2\}$ needs to be considered, for which $a = 1$ satisfies the three conditions. Similarly, the point $x = 3$ is a limit point. Finally, the point $x = 4$ is not a limit point because by choosing $x \in B = \{4\}$, the conditions $a \in A$ and $a \in B$ are contradictory. Thus we conclude $\{1\}' = \{2, 3\}$.

The limit points of $\{2\}$, $\{3\}$, $\{4\}$ are all empty.

**Example 4.5.2.** What are the limit points of $(0, 1)$ with respect to some of the topological bases in Example 4.1.3?

In the usual topology $\mathbb{R}_{(\leftrightarrow)}$, we have $(0, 1)'_{(\leftrightarrow)} = [0, 1]$.

In the lower limit topology $\mathbb{R}_{[\leftrightarrow)}$, we need to consider $x \in B = [a, b)$. Since $x \in [x, b) \subset [a, b)$, it is sufficient to consider $x \in B = [x, b)$ only. If $0 \leq x < 1$, then $((0, 1) - x) \cap B = (x, \min\{1, b\}) \neq \emptyset$ (see the middle part of Figure 4.5.1). Therefore $x$ is a limit point. If $x < 0$, then $x \in B = [x, 0)$ and $B$ contains no point in $(0, 1)$ (see the left part of Figure 4.5.1). Similarly, if $x \geq 1$, then $x \in B = [x, x + 1)$ and $B$ contains no point in $(0, 1)$ (see the right part of Figure 4.5.1). In either case, the first condition in the definition of limit points is not satisfied, so that $x$ is not a limit point. In conclusion, we get $(0, 1)'_{[\leftrightarrow)} = [0, 1)$.

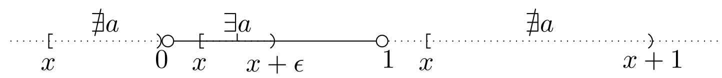



**Figure 4.5.1.** *limit points of* $(0, 1)$ *in the lower limit topology.*

Finally, consider the topology $\mathbb{R}_{(\leftarrow}$. If $x > 1$, then $x \in B = (1, \infty)$ and $B$ contains no point in $(0, 1)$. Therefore the first condition is not satisfied, and $x$ is not a limit point. If $x \leq 1$, then for any $x \in B = (a, \infty)$, we have $a < x \leq 1$. This implies that $((0, 1) - x) \cap B \supset (b, 1) - x$, where $b = \max\{a, 0\}$. Since $(b, 1) - x$ is never empty, $x$ is a limit point. We conclude that $(0, 1)'_{(\leftarrow} = (-\infty, 1]$.

**Example 4.5.3.** Consider the pointwise convergence topology in Examples 4.1.5 and 4.4.8. We claim that any function in $C[0, 1]$ is a limit point of the unit ball in the $L_1$-metric

$$A = \left\{ f \in C[0, 1] \colon d_{L_1}(f, 0) = \int_0^1 |f(t)| dt < 1 \right\}.$$

In other words, for any $g \in B = B(a_1, \ldots, a_n, t_1, \ldots, t_n, \epsilon)$, we need to find $f$ in $(A - g) \cap B$. Pick any $t_0 \in [0, 1]$ distinct from $t_1, \ldots, t_n$. Then it is easy to construct (see Figure 4.5.2) a continuous function $f$ satisfying

$$f(t_0) \neq g(t_0);\ f(t_1) = g(t_1),\ \ldots,\ f(t_n) = g(t_n);\ \int_0^1 |f(t)| dt < 1.$$

The first inequality implies $f \neq g$. The equalities in the middle imply $f \in B$. The last inequality implies $f \in A$.

Exercise 4.5.1. Prove that any subset has no limit point in the discrete topology.

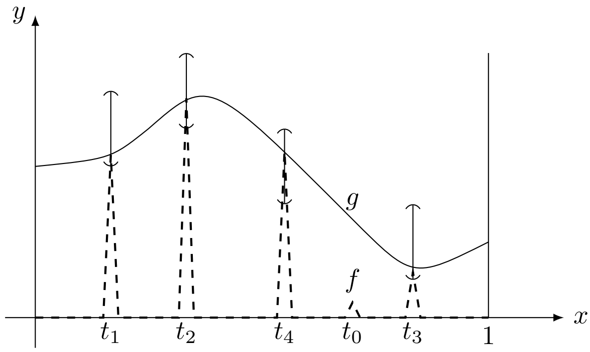



**Figure 4.5.2.** *$f$ is equal to $g$ at finitely many points but has small integration.*

Exercise 4.5.2. Prove that in the trivial topology, if $A$ contains at least two points, then $A' = X$. Moreover, $\{a\}' = X - a$, $\emptyset' = \emptyset$.

Exercise 4.5.3. Prove that in the finite complement topology, $A' = X$ or $\emptyset$ according to whether $A$ is infinite or finite.

Exercise 4.5.4. Prove that in the lower limit topology, $x$ is a limit point of $A$ if and only if there is a strictly decreasing sequence in $A$ converging to $x$ in the usual sense.

Exercise 4.5.5. Prove that the limit points of any subset $A \subset \mathbb{R}$ with respect to the topology $\mathbb{R}_{\leftarrow}$ are

$$A'_{\leftarrow} = \begin{cases} \mathbb{R}, & \text{if } \sup A = +\infty, \\ (-\infty, \sup A], & \text{if } (\sup A - \epsilon, \sup A) \cap A \neq \emptyset \text{ for any } \epsilon > 0, \\ (-\infty, \sup A), & \text{if } (\sup A - \epsilon, \sup A) \cap A = \emptyset \text{ for some } \epsilon > 0. \end{cases}$$

Exercise 4.5.6. Prove that in the Michael line in Exercise 4.4.6, the limit points of any subset $A$ are the rational limits of $A$ in the usual topology: $A'_{\text{Michael}} = A'_{\leftrightarrow} \cap \mathbb{Q}$. What about the more general topology in Exercise 4.4.6?

Exercise 4.5.7. Find the limit points of the unit disk $\{(x, y) \colon x^2 + y^2 < 1\}$ with respect to the topological bases in Example 4.1.4.

Exercise 4.5.8. Find the limit points of the subsets $\{n\}$, $n\mathbb{N}$, and $1 + 2\mathbb{N}$ (all odd numbers) of $\mathbb{N}$.

1. Topological subbasis $\{p\mathbb{N} \colon p \text{ is a prime number}\}$ in Exercise 4.1.9.
2. Topology $\{n\mathbb{N} \colon n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$ in Exercise 4.3.1.
3. Topology $\{2^n\mathbb{N} \colon n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$ in Exercise 4.3.1.

Exercise 4.5.9. Find the limit points of the subsets $[0, 1]$, $(0, 1)$, $(\sqrt{2}, \sqrt{10})$, $\mathbb{Z}$, and $(-\infty, 0)$ of $\mathbb{R}$.

1. Topological basis $\{(a,b)\colon a,b\in\mathbb{Z}\}$.
2. Topological basis $\{[a,b)\colon a\in\mathbb{Z}\}$.
3. Topological basis $\{[a,b)\colon b\in\mathbb{Z}\}$.
4. Topology $\{(-\infty,-a)\cup(a,\infty)\colon a>0\}\cup\{\emptyset,\mathbb{R}\}$.
5. Topology $\{(-a,2a)\colon \text{all } a>0\}\cup\{\emptyset,\mathbb{R}\}$.
6. Topological basis $\{(a_n,\infty)\colon n\in\mathbb{N}\}\cup\{\mathbb{R}\}$, where $a_n$ is an increasing sequence.

Exercise 4.5.10. The following conditions describe subsets of $C[0,1]$. Which have the property that any function is a limit point in the pointwise convergence topology?

1. $f(0)\geq 1$.
2. $f(0)\geq f(1)$.
3. $f(0)=0$.
4. $f(t)>0$ for any $0\leq t\leq 1$.
5. $\displaystyle\int_0^1|f(t)|dt\geq 1$.
6. $\displaystyle\int_0^1|f(t)|dt\geq 1$ and $f(0)\in\mathbb{Q}$.
7. $f(r)\in\mathbb{Q}$ for any $r\in[0,1]\cap\mathbb{Q}$.

Exercise 4.5.11. Prove that the open subset $U$ in the definition of limit points may be replaced by open subsets in a local topological basis (see Exercise 4.3.11) at $x$. The idea is already used in Example 4.5.2. Moreover, can the open subsets be replaced by subsets in a topological subbasis?

Exercise 4.5.12. Suppose $x$ is a limit point of $A$. Is $x$ still a limit point of $A$ in a coarser topology? What about in a finer topology?

Exercise 4.5.13. Extend the properties of limit points in Exercise 2.5.4 to general topological spaces:

$$A\subset B \implies A'\subset B',\quad (A\cup B)'=A'\cup B',\quad (A\cap B)'\subset A'\cap B'.$$

The relation between $A''$ and $A'$ will be answered in Exercise 4.5.22.

Exercise 4.5.14. Prove that if $x$ is a limit point of $A$, then $x$ is also a limit point of $A-x$.

Exercise 4.5.15. Let $x$ be a limit point of a subset $A$. Let $U$ be an open subset containing $x$. Prove that $x$ is a limit point of $A\cap U$. However, since finite subsets of topological spaces can have limit points (see Example 4.5.1), the second part of Exercise 2.5.8 cannot be extended to general topological spaces?

Exercise 4.5.16. We say a sequence $\{a_n\colon n\in\mathbb{N}\}$ has limit $x$ and denote $\lim a_n=x$, if for any $x\in U$, $U$ open, there is $N$, such that $n>N$ implies $a_n\in U$. The following extends Exercise 2.5.5.

1. Prove that if $x$ is the limit of a sequence $a_n\in A$ satisfying $a_n\neq x$, then $x$ is a limit point of $A$.
2. Prove that if $x$ has a countable local topological basis (see Exercise 4.3.12) and is a limit point of $A$, then $x=\lim a_n$ for a sequence $a_n\in A$ satisfying $a_n\neq x$.

Moreover, explain why the countability assumption in the second part is necessary.

Proposition 2.6.2 suggests the following definition of closed subsets in a topological space.

**Definition 4.5.1.** A subset $C$ of a topological space $X$ is *closed* if $X - C$ is open.

The proof of Proposition 2.6.2 is still valid by replacing $B(x, \epsilon)$ with open $U$. Therefore a subset is closed if and only if it contains all its limit points.

By $X - (X - U) = U$, it is easy to see that $U \subset X$ is open if and only if the complement $X - U$ is closed. By $f^{-1}(Y - B) = X - f^{-1}(B)$, we also know that a map is continuous if and only if the preimage of closed subsets are closed. Moreover, by de Morgan's law, Proposition 4.2.2 may be converted to the following properties of closed subsets (compare Corollary 2.6.4).

**Proposition 4.5.2.** *The closed subsets of a topological space $X$ satisfy the following:*

1. *$\emptyset$ and $X$ are closed.*
2. *Intersections of closed subsets are closed.*
3. *Finite unions of closed subsets are closed.*

**Example 4.5.4.** In the discrete topology, all subsets are open and closed. In the trivial topology, the only closed subsets are the empty set and the whole space.

**Example 4.5.5.** The closed subsets in the finite complement topology in Example 4.3.6 are the finite subsets and the whole space.

**Example 4.5.6.** By Example 4.5.2, $(0,1)$ does not contain all its limit points with respect to any of $\mathbb{R}_{(\leftrightarrow)}$, $\mathbb{R}_{[\leftrightarrow)}$, $\mathbb{R}_{(\leftarrow}$. Thus $(0,1)$ is not closed with respect to any of the three topologies.

**Example 4.5.7.** We would like to find the coarsest topology on $X = \{1,2,3,4\}$, such that the subsets $\{1,2,3\}$, $\{3,4\}$, $\{2,4\}$ are closed. The requirement is the same as that the subsets $\{4\}$, $\{1,2\}$, $\{1,3\}$ are open. The answer is the topology in Example 4.3.5.

**Example 4.5.8.** We expect the subset

$$A = \{f \colon f(t) \geq 0 \text{ for all } 0 \leq t \leq 1\}$$

of all non-negative functions to be closed under the pointwise convergence topology because $f_n(t) \geq 0$ for any $t, n$ and $\lim f_n(t) = f(t)$ for any $t$ imply $f(t) \geq 0$.

To actually prove that $A$ is open, we consider the complement $U = C[0,1] - A$, which consists of all functions are are negative somewhere. Suppose $f \in U$. Then $f(t_0) < 0$ for some $t_0 \in [0,1]$. Since

$$\begin{aligned} g \in B(f(t_0), t_0, -f(t_0)) &\implies |g(t_0) - f(t_0)| < -f(t_0) \\ &\implies 2f(t_0) < g(t_0) < 0 \\ &\implies g \in U, \end{aligned}$$

we get $f \in B(f(t_0), t_0, -f(t_0)) \subset U$. Therefore $U$ is open in the pointwise convergence topology.

Since the $L_\infty$-topology is finer than the pointwise convergence topology, the subset $A$ is also $L_\infty$-closed. Another way to see the closedness of $A$ is that $A = \cap_{t\in[0,1]} E_t^{-1}[0, +\infty)$ and that the evaluation maps $E_t$ are continuous with respect to the pointwise convergence and $L_\infty$-topologies.

Exercise 4.5.17. By saying $[a, b]$ is a closed interval, we really mean it is closed with respect to the usual topology. Study whether $[a, b]$ is still closed with respect to the topological bases in Example 4.1.3.

Exercise 4.5.18. Which subsets in Exercise 4.5.10 are closed in the pointwise convergence topology?

Exercise 4.5.19. Find the coarsest topology on $X = \{1, 2, 3, 4\}$ such that $\{1, 2\}$, $\{3, 4\}$ are open and $\{1, 2\}$, $\{2, 3\}$ are closed.

Exercise 4.5.20. Suppose $U$ is open and $C$ is closed. Prove that $U - C$ is open and $C - U$ is closed.

Exercise 4.5.21. Prove that the finite complement topology is the coarsest topology such that any single point subset is closed.

Exercise 4.5.22. A topological space is *Fréchet*[8] if any single point subset is closed.

1. Prove that $A'' \subset A'$ for any subset $A$ in a Fréchet space.
2. Construct an example showing that $A''$ may not be contained in $A'$ in general.

## 4.6 Closure

Since being closed is equivalent to containing all the limit points, it is natural to expect that, by adding all the limit points to any subset, one should get a closed subset. The result is the *closure*, given by the following equivalent descriptions.

**Lemma 4.6.1.** *For any subset $A$, the following characterize the same subset $\bar{A}$.*

1. $\bar{A} = A \cup A'$.
2. $\bar{A} = \cap\{C : C$ *is closed, and* $A \subset C\}$, *the smallest closed subset containing $A$.*

Note that the second description gives the smallest closed subset because the intersection of closed subsets is closed.

*Proof.* We prove the first subset is contained in the second one. This means that if $A \subset C$ for a closed $C$, then $A' \subset C$. Let $x$ be a limit point of $A$ and let $U$ be an open neighborhood of $x$. Then we have $(C - x) \cap U \supset (A - x) \cap U \neq \emptyset$. This proves that $x$ is also a limit point of $C$. Since $C$ is closed, we conclude that $x \in C$.

[8]Maurice René Fréchet, born September 2, 1878 in Maligny, France, died June 4, 1973 in Paris, France. Fréchet introduced the concepts of metric spaces and compactness in 1906. He made major contributions to topology, analysis and statistics.

Conversely, to prove the second subset is contained in the first one is the same as showing $A \cup A'$ is closed. Note that

$$\begin{aligned} x \in X - A \cup A' &\iff x \notin A,\ x \notin A' \\ &\iff x \notin A,\ x \in U,\ (A - x) \cap U = \emptyset \text{ for some open } U \\ &\iff x \notin A,\ x \in U,\ A \cap U = \emptyset \text{ for some open } U \\ &\iff x \in U,\ A \cap U = \emptyset \text{ for some open } U. \end{aligned}$$

For the open subset $U$ found in the argument above, we have

$$y \in U \implies (A - y) \cap U \subset A \cap U = \emptyset.$$

This shows that points in $U$ are not limit points of $A$, or $A' \cap U = \emptyset$. Combined with the conclusion $A \cap U = \emptyset$, we get $x \in U \subset X - A \cup A'$. This proves that $X - A \cup A'$ is open. $\square$

The proof of the lemma tells us that $x \neq \bar{A}$ if and only if $A \cap U = \emptyset$ for some open neighborhood $U$ of $x$. This is equivalent to the following useful characterization of points in the closure.

**Lemma 4.6.2.** *A point $x$ is in the closure of $A$ if and only if $A \cap U \neq \emptyset$ for any open neighborhood $U$ of $x$. Moreover, $U$ may be replaced by subsets in a topological basis.*

A subset of a topological space is *dense* if its closure is the whole space. For example, the rational numbers are dense among the real numbers in the usual topology as well as the lower limit topology. The lemma tells us that a subset $A$ is dense if and only if $A \cap U \neq \emptyset$ for any nonempty open subset $U$. This is also the same as the complement $X - A$ not containing any nonempty open subset. Moreover, the nonempty open subsets may be replaced by subsets in a topological basis.

**Example 4.6.1.** In the topological space in Example 4.5.1, the closure of $\{1\}$ is $\{1\} \cup \{2, 3\} = \{1, 2, 3\}$. Alternatively, the topology $\mathcal{T}$ in Example 4.3.5 gives us all the closed subsets (by taking complements)

$$\{1,2,3,4\}, \{2,3,4\}, \{1,2,3\}, \{3,4\}, \{2,4\}, \{2,3\}, \{4\}, \{3\}, \{2\}, \emptyset.$$

Among these, $\{1, 2, 3\}$ is the smallest that contains $\{1\}$.

**Example 4.6.2.** By Example 4.5.2, the closure of the interval $(0, 1)$ with respect various topologies are

$$\overline{(0,1)}_{(\leftrightarrow)} = [0,1], \quad \overline{(0,1)}_{[\mapsto)} = [0,1), \quad \overline{(0,1)}_{(\leftarrow} = (-\infty,1].$$

**Example 4.6.3.** In a metric space, the distance $d(x, A)$ from a point $x$ to a subset $A$ is introduced in Exercise 2.4.15. By Exercise 2.5.10 and the first description in Lemma 4.6.1, we have

$$\bar{A} = \{x \in X : d(x, A) = 0\}.$$

The equivalence between $x \in \bar{A}$ and $d(x,A) = 0$ can also be proved by making use of Lemma 4.6.2. Note that the lemma still holds if $U$ is replaced by balls $B(x,\epsilon)$. Then $x \in \bar{A}$ means $A \cap B(x,\epsilon) \neq \emptyset$ for any $\epsilon > 0$. However, $a \in A \cap B(x,\epsilon)$ means $a \in A$ and $d(x,a) < \epsilon$. Thus the nonemptyness simply means the distance between $x$ and points of $A$ is smaller than any $\epsilon > 0$. This means exactly $d(x,A) = 0$.

**Example 4.6.4.** Example 2.5.5 tells us that the polynomials is $L_\infty$-dense in $C[0,1]$. In fact, polynomials are also $L_1$-dense as well as dense in the pointwise convergence topology. (You will see the reason after doing Exercise 4.6.7.)

Example 4.5.3 tells us that the $L_1$-unit ball $A$ is dense in pointwise convergence topology. However, Exercise 2.5.7 implies that both the $L_1$-closure and the $L_\infty$-closure of $A$ is the closed $L_1$-unit ball

$$\bar{B}_{L_1}(0,1) = \left\{ f \in C[0,1] \colon d_{L_1}(f,0) = \int_0^1 |f(t)|dt \le 1 \right\}.$$

Therefore the $L_1$-unit ball $A$ is not dense in $L_1$- or the $L_\infty$-topology.

Exercise 4.6.1. Find the closures of the subsets in various topologies in Exercises 4.5.5 through 4.5.10.

Exercise 4.6.2. Find the closure of the *topologist's sine curve*

$$\Sigma = \left\{ \left( x, \sin\frac{1}{x} \right) : x > 0 \right\}$$

in the usual topology of $\mathbb{R}^2$.

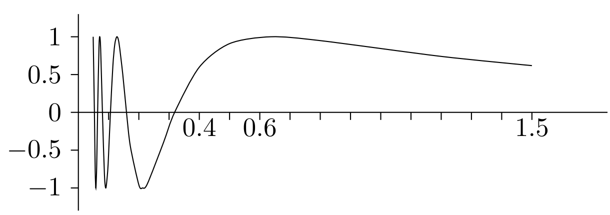



**Figure 4.6.1.** *topologist's sine curve*

Exercise 4.6.3. For integers $m \le n$, denote

$$[m,n] = \{k \in \mathbb{Z} \colon m \le k \le n\}, \quad n\mathbb{Z} = \{kn \colon k \in \mathbb{Z}\}.$$

Find the closure of any subset of $\mathbb{Z}$ with respect to the following topological bases.

1. $\{[-n,n] \colon n \ge 0\}$.
2. $\{[2m,3n] \colon 2m \le 3n, m \in \mathbb{Z}, n \in \mathbb{Z}\}$.
3. $\{n\mathbb{Z} \colon n \in \mathbb{Z}\}$.
4. $\{2^n\mathbb{Z} \colon n \in \mathbb{Z}\}$.

Exercise 4.6.4. Is it always true that the closed ball $\bar{B}(a,\epsilon) = \{x \colon d(x,a) \le \epsilon\}$ in a metric space is the closure of the open ball $B(a,\epsilon)$?

Exercise 4.6.5. Prove that any dense subset of the Michael line must contain all the irrational numbers.

Exercise 4.6.6. Suppose $A$ is a dense subset of a metric space $X$. Prove that all the balls $B(a,\epsilon)$ with center $a \in A$ form a topological basis of $X$.

Exercise 4.6.7. How is the closure changed if the topology is made coarser or finer?

Exercise 4.6.8. Prove that $U$ is open if and only if $\bar{A} \cap U \subset \overline{A \cap U}$ for any subset $A$. This is also equivalent to $\overline{U \cap \bar{A}} = \overline{U \cap A}$ for any $A$.

Exercise 4.6.9. Prove properties of the closure.

1. $A$ is closed $\iff$ $A = \bar{A}$.
2. $A \subset B \implies \bar{A} \subset \bar{B}$.
3. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
4. $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.
5. $\bar{\bar{A}} = \bar{A}$.

Exercise 4.6.10 (Kuratowski). Prove that the axioms for topology can be rephrased in terms of the closure. In other words, a topology on $X$ may be defined as an operation $A \mapsto \bar{A}$ on subsets of $X$ satisfying

- $\bar{\emptyset} = \emptyset$.
- $\overline{\{x\}} = \{x\}$.
- $\bar{\bar{A}} = \bar{A}$.
- $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Exercise 4.6.11. The *interior* $\mathring{A}$ of a subset $A$ is the biggest open subset contained in $A$. The concept is complementary to the concept of closure.

1. Prove that $\mathring{A} = X - \overline{X - A}$.
2. Prove that $x \in \mathring{A}$ if and only if there is an open subset $U$, such that $x \in U \subset A$. In other words, the interior $\mathring{A}$ consists of all the points for which $A$ is a neighborhood. Moreover, the open subset $U$ may be replaced by subsets in a topological basis.
3. Study the properties of $\mathring{A}$ (similar to Exercises 4.6.7 to 4.6.9).
4. Rephrase the axioms of topology in terms of the properties of the interior operation.

Exercise 4.6.12. The *boundary* of a subset $A$ is $\partial A = \bar{A} \cap \overline{X - A}$.

1. Prove that $x \in \partial A$ if and only if any open neighborhood of $x$ contains points in $A$ as well as points outside $A$.
2. Prove that $X = \mathring{A} \sqcup \partial A \sqcup (X - A)^{\circ}$, $\bar{A} = \mathring{A} \sqcup \partial A$ and $\mathring{A} = A - \partial A$.
3. Prove that $A$ is closed if and only if $\partial A \subset A$, and $A$ is open if and only if $A \cap \partial A = \emptyset$.
4. Study the properties of $\partial A$.

Exercise 4.6.13 (Kuratowski 14-Set Theorem). Applying the closure and the complement operations alternatively to a subset $A \subset X$ produces subsets $A$, $X - A$, $\bar{A}$, $\overline{X - A}$, $X - \bar{A}$, $X - \overline{X - A}$, $\dots$.

1. Prove that if $A$ is the closure of an open subset, then $A = \overline{X - \overline{X - A}}$. In other words, $A$ is the closure of its interior.
2. Prove that for any $A$, the two operations produce at most 14 distinct subsets.
3. Find a subset $A$ of $\mathbb{R}_{(-)}$ for which the maximal number 14 is obtained.

Chapter 5

# Basic Topological Concepts

## 5.1 Continuity

Proposition 2.4.2 suggests the following extension of continuity to maps between topological spaces.

**Definition 5.1.1.** A map $f\colon X \to Y$ between topological spaces is *continuous* if

$$U \subset Y \text{ is open} \implies f^{-1}(U) \subset X \text{ is open}.$$

**Example 5.1.1.** Any map *from* a discrete topology is continuous. Any map *to* a trivial topology is continuous.

**Example 5.1.2.** For a fixed $b \in Y$, the preimage of $U \subset Y$ under the constant map $f(x) = b\colon X \to Y$ is

$$f^{-1}(U) = \begin{cases} X, & \text{if } b \in U, \\ \emptyset, & \text{if } b \notin U. \end{cases}$$

Therefore $f^{-1}(U)$ is always an open subset of $X$, and $f$ is continuous.

**Example 5.1.3.** The preimages of $U \subset \mathcal{T}$ under the maps

$$\rho(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases} : \mathbb{R}_{[\,)} \to \mathbb{R}_{\mathcal{T}}, \quad \theta(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases} : \mathbb{R}_{[\,)} \to \mathbb{R}_{\mathcal{T}}$$

are

$$\rho^{-1}(U) = \begin{cases} \emptyset, & \text{if } 0, 1 \notin U, \\ [0, \infty), & \text{if } 0 \notin U, 1 \in U, \\ (-\infty, 0), & \text{if } 0 \in U, 1 \notin U, \\ \mathbb{R}, & \text{if } 0, 1 \in U, \end{cases} \quad \theta^{-1}(U) = \begin{cases} \emptyset, & \text{if } 0, 1 \notin U, \\ (0, \infty), & \text{if } 0 \notin U, 1 \in U, \\ (-\infty, 0], & \text{if } 0 \in U, 1 \notin U, \\ \mathbb{R}, & \text{if } 0, 1 \in U. \end{cases}$$

Therefore $\rho$ is continuous regardless of what $\mathcal{T}$ is, and $\theta$ is continuous if and only if $\mathcal{T}$ does not contain any subset $U$ satisfying $0 \in U$ and $1 \notin U$.

The continuity can be verified by making use of topological basis or subbasis.

**Lemma 5.1.2.** *The following are equivalent for a map $f\colon X \to Y$ between topological spaces.*

1. *$f$ is continuous.*
2. *$f^{-1}(B)$ is open for any $B$ in a topological basis of $Y$.*
3. *$f^{-1}(S)$ is open for any $S$ in a topological subbasis of $Y$.*

The first and the second are equivalent because by Lemma 4.2.3, open subsets are of the form $U = \cup B_i$ with $B_i$ in the topological basis, and $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$. The second and the third are equivalent because the subsets in the induced topological basis are of the form $B = S_1 \cap \cdots \cap S_n$ with $S_i$ in the topological subbasis, and $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

Note that all the discussions about continuous maps between metric spaces are still valid, and the same proofs apply. For example, a map is continuous if and only if the preimage of any closed subset is closed, and compositions of continuous maps are continuous.

**Example 5.1.4.** By Example 2.4.4, the addition map $\sigma(x,y) = x+y\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, provided both sides have the usual topology. Is $\sigma$ still continuous as a map $\mathbb{R}^2_{[\square)} \to \mathbb{R}_{[\,)}$ or $\mathbb{R}^2_{[\boxdot)} \to \mathbb{R}_{[\,)}$?

By Lemma 5.1.2, the key point is whether $\sigma^{-1}[a,b)$ is open. The preimage is shown on the left of Figure 5.1.1, which is easily seen to be open in $\mathbb{R}^2_{[\square)}$ but not open in $\mathbb{R}^2_{[\boxdot)}$. Therefore $\sigma\colon \mathbb{R}^2_{[\square)} \to \mathbb{R}_{[\,)}$ is continuous, but $\sigma\colon \mathbb{R}^2_{[\boxdot)} \to \mathbb{R}_{[\,)}$ is not continuous.

We may also consider the subtraction map $\delta(x,y) = x - y\colon \mathbb{R}^2 \to \mathbb{R}$. If $\mathbb{R}$ has the lower limit topology, then the continuity of $\delta$ means the openness of $\delta^{-1}[a,b)$. The preimage is shown on the right of Figure 5.1.1, which suggests that both $\delta\colon \mathbb{R}^2_{[\square)} \to \mathbb{R}_{[\,)}$ and $\delta\colon \mathbb{R}^2_{[\boxdot)} \to \mathbb{R}_{[\,)}$ are not continuous.

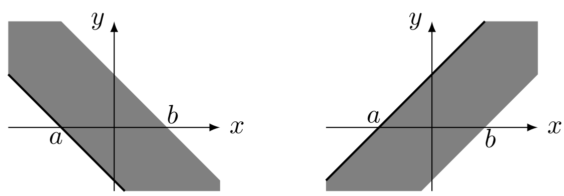



**Figure 5.1.1.** $\sigma^{-1}[a,b)$ *and* $\delta^{-1}[a,b)$

**Example 5.1.5.** By Example 2.4.2, the evaluation map $E(f) = f(0)\colon C[0,1]_{L_\infty} \to \mathbb{R}_{(\,)}$ is continuous. What if the $L_\infty$-topology is changed to the pointwise convergence topology?

Since $\mathbb{R}_{(\,)}$ has a subbasis consisting of $(a,\infty)$ and $(-\infty,a)$, by Lemma 5.1.2, it is sufficient to consider the openness of $E^{-1}(a,\infty)$ and $E^{-1}(-\infty,a)$. Since

$$E^{-1}(a,\infty) = \{f \in C[0,1]\colon E(f) \in (a,\infty)\} = \{f \in C[0,1]\colon f(0) > a\},$$

by an argument similar to Example 4.2.6, we find $E^{-1}(a,\infty)$ to be open in the pointwise convergence topology. Similarly, $E^{-1}(-\infty,a)$ is open. Therefore the evaluation map is continuous if $C[0,1]$ has the pointwise convergence topology.

Exercise 5.1.1. Find all the continuous maps from $\mathbb{R}_{(\,)}$ to the space $\{1,2\}$ equipped with the topology $\big\{\emptyset,\{1\},\{1,2\}\big\}$.

Exercise 5.1.2. Prove that in each of the following cases, the only continuous maps from $\mathbb{R}$ to $\mathbb{R}$ are the constants.

1. From the finite complement topology to the usual topology.
2. From the usual topology to the lower limit topology.
3. From the usual topology to the Michael line in Exercise 4.4.6.

For the last two problems, you may us the last conclusion of Exercise 2.6.1.

Exercise 5.1.3. Study the continuity of the multiplication map $\mu(x, y) = xy \colon \mathbb{R}^2 \to \mathbb{R}_{[\to)}$, where $\mathbb{R}^2$ is either $\mathbb{R}^2_{\square}$ or $\mathbb{R}^2_{\square}$.

Exercise 5.1.4. Which of the functions $2t$, $\frac{1}{2}t$, $t^2$ are continuous with respect to the topology on $\mathbb{R}$ induced by the topological basis $\{(a, b) \colon a \in \mathbb{Z}\}$.

Exercise 5.1.5. Find integers $a$ and $b$, such that the function $at + b$ from $\mathbb{Z}$ to itself is continuous with respect to the topological bases in Exercise 4.6.3.

Exercise 5.1.6. Find $a$, such that the function $f(x) = x + a$ is continuous for the Michael line in Exercise 4.4.6?

Exercise 5.1.7. Prove that the coarsest topology on $\mathbb{R}^2$ such that the addition $\sigma(x, y) = x+y$ and the subtraction $\delta(x, y) = x-y$ are continuous maps to $\mathbb{R}_{(\leftrightarrow)}$ is the usual topology. What about the coarsest topology such that the addition $\sigma$ and the multiplication $\mu(x, y) = xy$ are continuous?

Exercise 5.1.8. Find the coarsest topology $\mathcal{T}$ on $\mathbb{R}$ such that the functions

$$\rho_a(t) = \begin{cases} 1, & \text{if } t \geq a \\ 0, & \text{if } t < a \end{cases} \colon \mathbb{R}_{\mathcal{T}} \to \mathbb{R}_{(\leftrightarrow)}$$

are continuous for all $a$.

Exercise 5.1.9. Find the coarsest topology $\mathcal{T}$ on $\mathbb{R}$ such that the function $t^2 \colon \mathbb{R}_{\mathcal{T}} \to \mathbb{R}_{(\leftrightarrow)}$ is continuous.

Exercise 5.1.10. Find the coarsest topology $\mathcal{T}$ on $\mathbb{R}$ such that $(0, \infty)$ is open and the function $\frac{1}{2}t - 1 \colon \mathbb{R}_{\mathcal{T}} \to \mathbb{R}_{\mathcal{T}}$ is continuous.

Exercise 5.1.11. Prove that the pointwise convergence topology is the coarsest topology such that the evaluation maps $E_t(f) = f(t) \colon C[0, 1] \to \mathbb{R}_{(\leftrightarrow)}$ are continuous for any $t$.

Exercise 5.1.12. Let $\mathbb{R}$ have the topology $\{(a, \infty) \colon \text{all } a\} \cup \{\emptyset, \mathbb{R}\}$. Let $f \colon X = \{1, 2, 3, 4\} \to \mathbb{R}$ be the map that takes the point $x \in X$ to the number $x \in \mathbb{R}$. Find the coarsest topology on $X$ such that $\{1\}$ is open, $\{2, 3\}$ is closed, and $f$ is continuous. Then find the limit points and the closure of $\{4\}$ in the topology.

Exercise 5.1.13. In view of Example 5.1.1, answer the following for a topological space.

1. If any map from $X$ is continuous, is $X$ necessarily discrete?
2. If any map to $X$ is continuous, is $X$ necessarily trivial?

Exercise 5.1.14. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on $X$. Prove that the identity map $id \colon X_{\mathcal{T}} \to X_{\mathcal{T}'}$ is continuous if and only if $\mathcal{T}'$ is coarser than $\mathcal{T}$.

Exercise 5.1.15. Let $f \colon X \to Y$ be a continuous map. Will $f$ still be continuous if the topologies on $X$ or $Y$ are changed to finer or coarser ones?

Exercise 5.1.16. Let $f: X \to Y$ be a map. Given a topology on $Y$, find the coarsest topology on $X$ such that $f$ is continuous. On the other hand, given a topology on $X$, can you find coarsest or finest topology on $Y$ so that $f$ is continuous?

Given topological bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for $X$ and $Y$, we look at the second description in Lemma 5.1.2 in more detail. The openness of $f^{-1}(B)$ for $B \in \mathcal{B}_Y$ means

$$x \in f^{-1}(B) \text{ for some } B \in \mathcal{B}_Y \implies x \in B' \subset f^{-1}(B) \text{ for some } B' \in \mathcal{B}_X.$$

This leads to the following criterion of continuity in terms of topological bases.

**Lemma 5.1.3.** *Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be topological bases for $X$ and $Y$. Then a map $f: X \to Y$ is continuous if and only if*

$$f(x) \in B \textit{ for some } B \in \mathcal{B}_Y \implies x \in B' \textit{ and } f(B') \subset B \textit{ for some } B' \in \mathcal{B}_X.$$

The criterion is quite similar to the $\epsilon$-$\delta$ criterion for continuous maps between metric spaces. Also note that the third description in Lemma 5.1.2 leads to the following criterion for continuity:

$$f(x) \in S \text{ for some } S \in \mathcal{S}_Y \implies x \in B' \text{ and } f(B') \subset S \text{ for some } B' \in \mathcal{B}_X.$$

**Example 5.1.6.** By Lemma 5.1.3, the continuity of $f: \mathbb{R}_{[\,)} \to \mathbb{R}_{(\,)}$ means the following implication

$$f(x) \in (a,b) \implies \text{There is } [c,d), \text{ such that } x \in [c,d) \text{ and } f[c,d) \subset (a,b).$$

We will show that this is equivalent to that the function is *right continuous*: For any $x$ and $\epsilon > 0$, there is $\delta > 0$, such that

$$x \le y < x + \delta \implies |f(y) - f(x)| < \epsilon.$$

Given the implication for the continuity of $f$, for any $\epsilon > 0$, take $(a,b) = (x-\epsilon, x+\epsilon)$. Then there is $[c,d)$ satisfying $x \in [c,d)$ and $f[c,d) \subset (f(x)-\epsilon, f(x)+\epsilon)$. For $\delta = d-x > 0$, we also have $[x, x+\delta) = [x,d) \subset [c,d)$, and $f[x,x+\delta) \subset f[c,d) \subset (f(x)-\epsilon, f(x)+\epsilon)$. The inclusion means that $x \le y < x+\delta$ implies $|f(y) - f(x)| < \epsilon$.

Conversely, suppose $f$ is right continuous at $x$. If $f(x) \in (a,b)$, then $(f(x)-\epsilon, f(x)+\epsilon) \subset (a,b)$ for some $\epsilon > 0$ (take $\epsilon = \min\{f(x)-a, b-f(x)\}$, for example). By the right continuity, there is $\delta > 0$, such that $f[x, x+\delta) \subset (f(x)-\epsilon, f(x)+\epsilon)$. Then we have $x \in [x, x+\delta)$ and $f[x,x+\delta) \subset (f(x)-\epsilon, f(x)+\epsilon) \subset (a,b)$. This verifies the implication for the continuity of $f$.

Exercise 5.1.17. Given toplogical bases on $X$ and $Y$, what do you need to do in order to show that a map $f: X \to Y$ is *not* continuous?

Exercise 5.1.18. Prove that $B$ and $B'$ in Lemma 5.1.3 may be replaced by subsets in local topological bases (see Exercise 4.3.11) or neighborhood systems (see Exercise 4.3.13) at $x$ and $f(x)$. The idea is already used in Example 5.1.6.

Exercise 5.1.19. What topology $\mathcal{T}$ has the property that the continuity of $f\colon \mathbb{R}_{\mathcal{T}} \to \mathbb{R}_{(\leftrightarrow)}$ means left continuous?

Exercise 5.1.20. What topology $\mathcal{T}$ has the property that the continuity of $f\colon \mathbb{R}^2_{\mathcal{T}} \to \mathbb{R}_{(\leftrightarrow)}$ means the continuity in $x$ variable (but not necessarily in $y$ variable)[9]?

Exercise 5.1.21. Prove that $f\colon \mathbb{R}_{[\leftrightarrow)} \to \mathbb{R}_{[\leftrightarrow)}$ is continuous if and only if the function is increasing and right continuous. Prove that if $f, g\colon \mathbb{R}_{[\leftrightarrow)} \to \mathbb{R}_{[\leftrightarrow)}$ are continuous, then $f+g$ is continuous. What about $f-g$?

Exercise 5.1.22. Determine the continuity.

1. $\displaystyle\int_0^1 f(t)dt\colon C[0,1]_{\text{pt conv}} \to \mathbb{R}_{(\leftrightarrow)}$.
2. $f^2\colon C[0,1]_{\text{pt conv}} \to C[0,1]_{\text{pt conv}}$.
3. $\max f\colon C[0,1]_{\text{pt conv}} \to \mathbb{R}_{(\leftrightarrow)}$.
4. $\max f\colon C[0,1]_{L_\infty} \to \mathbb{R}_{[\leftrightarrow)}$.

Exercise 5.1.23. Prove that $f$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$.

Exercise 5.1.24. Prove that $f$ is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

Exercise 5.1.25. Prove that if $f\colon X \to Y$ is continuous, then $\lim a_n = x$ implies $\lim f(a_n) = f(x)$ (see Exercise 4.5.16). Moreover, if any point in $X$ has a countable local topological basis (see Exercise 4.3.12), prove that the converse is also true.

Exercise 5.1.26. Let $f$ be a continuous map. If $x$ is a limit point of $A$, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Exercise 5.1.27. Let $f, g\colon X \to Y$ be continuous maps that are equal on a dense subset. Is it necessarily true that $f(x) = g(x)$ for all $x \in X$?

Exercise 5.1.28. A function $f\colon X \to \mathbb{R}$ is *upper semicontinuous* if $f^{-1}(a, \infty)$ is open for all $a$. It is *lower semicontinuous* if $f^{-1}(-\infty, a)$ is open for all $a$. Prove the following are equivalent.

1. $f$ is upper semicontinuous.
2. $-f$ is lower semicontinuous.
3. $f\colon X \to \mathbb{R}_{(\leftarrow}$ is continuous.
4. If $f(x) > a$, there there is a neighborhood $N$ of $x$, such that $f(y) > a$ for all $y \in N$.

Exercise 5.1.29. Prove that $f\colon X \to \mathbb{R}$ is continuous if and only if $f$ is upper and lower semicontinuous.

Exercise 5.1.30. Prove that if $f$ and $g$ are upper semicontinuous, then $f+g$ is also upper semicontinuous. What about $fg$?

[9] If you want $f$ to be continuous in $x$ variable, in $y$ variable, but not necessarily in both variables together, then you need to put "plus topology" on $\mathbb{R}^2$. See D. J. Velleman: *Multivariable Calculus and the Plus Topology*, AMS Monthly, vol.106, 733-740 (1999)

Exercise 5.1.31. Prove that the characteristic function $\chi_A$ is upper semicontinuous if and only if $A$ is closed, and the function is lower semicontinuous if and only if $A$ is open.

## 5.2 Homeomorphism

Among the four topologies on the two point space $X = \{1, 2\}$ in Example 4.3.4, $\mathcal{T}_2$ and $\mathcal{T}_3$ are interchanged via the map $f\colon X \to X, f(1) = 2, f(2) = 1$. The equivalences between some topologies on $X = \{1, 2, 3\}$ are also observed.

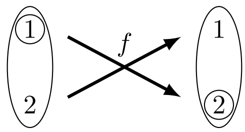



**Figure 5.2.1.** *homeomorphism between $\mathcal{T}_2$ and $\mathcal{T}_3$*

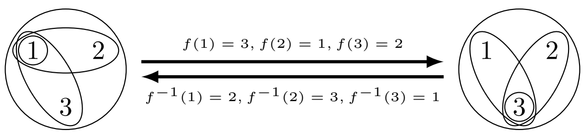



**Figure 5.2.2.** *homeomorphic topologies on three points*

**Definition 5.2.1.** A map $f\colon X \to Y$ between topological spaces is a *homeomorphism* if

- $f$ is invertible.
- Both $f$ and $f^{-1}$ are continuous.

Two topological spaces are *homeomorphic* if there is a homeomorphism between them.

A homeomorphism is an invertible map $f\colon X \to Y$ such that

$$U \subset X \text{ is open} \iff f(U) \subset Y \text{ is open}.$$

In other words, $f$ induces a one-to-one correspondence between the open subsets.

Homeomorphic spaces are considered to be the "same" from the topological viewpoint. In this regard, the concept of homeomorphism (called *isomorphism* in general) is a very universal and powerful one. For example, as far as making one trip across the seven bridges in Königsberg is concerned, the simple graph is isomorphic to the network of actual bridges. For another example, all finite dimensional vector spaces are isomorphic to the standard Euclidean space. The fact enables us to translate special properties about Euclidean spaces into general properties in linear algebra.

**Example 5.2.1.** The map $f(x) = 3x - 2 \colon \mathbb{R}_{(-)} \to \mathbb{R}_{(-)}$ is a homeomorphism because $f$ is invertible, and both $f(x)$ and $f^{-1}(y) = \frac{1}{3}y + \frac{2}{3}$ are continuous.

The map $f(x) = \dfrac{x}{1-x^2} \colon (-1,1)_{(-)} \to \mathbb{R}_{(-)}$ tells us that $(-1,1)$ and $\mathbb{R}$ are homeomorphic. Another homeomorphism between these two spaces is given by $g(x) = \tan \dfrac{\pi x}{2}$.

**Example 5.2.2.** The topological basis $\mathcal{B}_2$ in Example 4.1.3 induces the lower limit topology $\mathbb{R}_{[-)}$. Similarly, the topological basis $\mathcal{B}_3$ induces the *upper limit topology* $\mathbb{R}_{(-]}$. The map

$$f(x) = -x \colon \mathbb{R}_{[-)} \to \mathbb{R}_{(-]}$$

is a homeomorphism because $f$ gives a one-to-one correspondence between $\mathcal{B}_2$ and $\mathcal{B}_3$ and then gives a one-to-one correspondence between anything induced by the bases.

**Example 5.2.3.** Consider the identity map

$$id \colon X_{\mathcal{T}} \to X_{\mathcal{T}'}$$

between two topologies on $X$. The map $id$ is continuous if and only if $\mathcal{T}'$ is coarser than $\mathcal{T}$. The inverse map

$$id^{-1} = id \colon X_{\mathcal{T}'} \to X_{\mathcal{T}}$$

is continuous if and only if $\mathcal{T}$ is coarser than $\mathcal{T}'$. Therefore $id$ is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

**Example 5.2.4.** Consider the interval $[0,1)$ and the circle $S^1$, both with the usual topology. The restriction of the map $E(x) = e^{2\pi i x}$ to $[0,1)$ is invertible and continuous. However, $E$ is not a homeomorphism since $E^{-1}$ is not continuous: The subset $U$ in Figure 5.2.3 is open in $[0,1)$, but its image $E(U)$ under $E$ (which is the same as the preimage $(E^{-1})^{-1}(U)$ under $E^{-1}$) is not open in $S^1$.

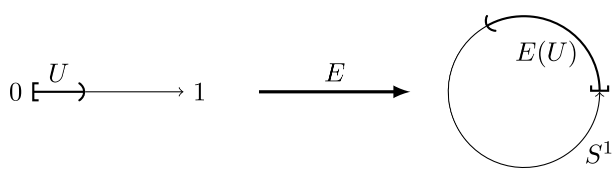



**Figure 5.2.3.** *Invertibility and continuity do not imply homeomorphism.*

Exercise 5.2.1. Let $X = \{1,2,3,4\}$ have the topology given by Example 4.3.5. Find all homeomorphisms from $X$ to itself.

Exercise 5.2.2. Describe all homeomorphisms from $\mathbb{R}_{(-)}$ to itself.

Exercise 5.2.3. Prove that all the open intervals, with the usual topology, are homeomorphic. What about closed intervals?

Exercise 5.2.4. Describe how the unit disk $\{(x, y) \colon x^2 + y^2 < 1\}$ and the unit square $(0, 1) \times (0, 1)$, with the usual topology, are homeomorphic.

Exercise 5.2.5. Let $X = \mathbb{R} \cup \{-\infty, +\infty\}$ be the straight line with two infinities. Let $X$ have the topology induced by the following topological basis

$$\mathcal{B} = \big\{(a, b)\big\} \cup \big\{(a, +\infty]\big\} \cup \big\{[-\infty, b)\big\},$$

where

$$(a, +\infty] = (a, +\infty) \cup \{+\infty\}, \quad [-\infty, b) = \{-\infty\} \cup (-\infty, b).$$

Prove that $X$ is homeomorphic to $[0, 1]$.

Exercise 5.2.6. Prove that $[0, 1]$ and $(0, 1)$ are not homeomorphic by showing that any continuous map $f \colon [0, 1] \to (0, 1)$ is not onto. Similarly, prove that $[0, 1)$, $[0, 1]$, $(0, 1)$ are not homeomorphic to each other.

Exercise 5.2.7. Prove that if $X$ and $Y$ are homeomorphic, and $Y$ and $Z$ are homeomorphic, then $X$ and $Z$ are homeomorphic.

Exercise 5.2.8. What is the condition for two discrete topological spaces to be homeomorphic? What about two trivial topological spaces?

Exercise 5.2.9. A map $f \colon X \to Y$ between metric spaces is an *isometry* if $f$ is invertible and $d_X(x, y) = d_Y(f(x), f(y))$. Prove that any isometry is a homeomorphism. Then find all the isometries from $\mathbb{R}$ (with the usual metric) to itself.

Since homeomorphisms induce one-to-one correspondences between open subsets, for any property (concept, theory) derived from open subsets, such a property (concept, theory) in one space exactly corresponds to the same property in the other one. For example, for a homeomorphism $f$ we have

$$x \text{ is a limit point of } A \iff f(x) \text{ is a limit point of } f(A).$$

**Definition 5.2.2.** A property $P$ about topological spaces is a *topological property* if

$$X \text{ and } Y \text{ are homeomorphic and } X \text{ has } P \implies Y \text{ has } P.$$

A quantity $I$ associated with topological spaces is a *topological invariant* if

$$X \text{ and } Y \text{ are homeomorphic} \implies I(X) = I(Y).$$

Any property described by open subsets is a topological property. For example, being a limit point is a topological property. The other topological properties include a subset being close, the closure of a subset, a topology being discrete. Other important topological properties to be studied are Hausdorff, connected, and compact. Among the simplest topological invariants is the Euler number, which will be studied later on. More advanced topological invariants include (covering) dimension, fundamental group, homology group, and homotopy group.

**Example 5.2.5.** A metric space is *bounded* if there is an upper bound for the distances between its points. With the usual metric, $\mathbb{R}$ is unbounded, while $(0,1)$ is bounded. Since $\mathbb{R}$ and $(0,1)$ are homeomorphic, the boundedness is a *metrical property* (a property preserved by isometries) but not a topological property.

**Example 5.2.6.** A topological space $X$ is *metrizable* if the topology can be induced by a metric on $X$. Metrizability is a topological property, because if $d$ induces the topology on $X$, and $f\colon Y \to X$ is a homeomorphism, then

$$d_Y(y_1, y_2) = d(f(y_1), f(y_2))$$

is a metric on $Y$ that induces the topology on $Y$.

**Example 5.2.7.** A topological space is *second countable* if it is induced by a topological basis consisting of countably many subsets. For example, the collection $\mathcal{B}_7$ in Example 4.1.3 is a countable basis for the usual topology on $\mathbb{R}$. Taking product of $\mathcal{B}_7$ with itself produces a countable basis for the usual topology on $\mathbb{R}^n$. So the usual topology is second countable.

If $\mathcal{B}$ is a (countable) topological basis for $X$ and $f\colon X \to Y$ is a homeomorphism, then $f(\mathcal{B}) = \{f(B)\colon B \in \mathcal{B}\}$ is also a (countable) topological basis for $Y$. Therefore the second countability is a topological property.

Let $\mathcal{B}$ be a topological basis for the lower limit topology. Since $[x,\infty)$ is open in the lower limit topology, there is some $B_x \in \mathcal{B}$, such that $x \in B_x \subset [x,\infty)$. This implies $x = \min B_x$, so that $B_x \neq B_y$ if $x \neq y$. Therefore the number of subsets in $\mathcal{B}$ is at least as many as the number of elements in $\mathbb{R}$, which is not countable. We conclude that the lower limit topology is not second countable and cannot be homeomorphic to the usual topology.

**Example 5.2.8.** A *local topological basis* at a point $x \in X$ is a collection $\mathcal{L}$ of open neighborhoods of $x$, such that

$$x \in U \text{ for some open } U \implies x \in L \subset U \text{ for some } L \in \mathcal{L}.$$

The concept is rather similar to topological basis. The only difference is that $x$ is fixed here. A topological space is *first countable* if any point has a countable local topological basis. Similar to the second countability, the first countability is also a topological property.

Although not second countable, the lower limit topology is first countable. For any strictly decreasing sequence $\epsilon_n$ with limit 0, the collection $\mathcal{L} = \{[x, x+\epsilon_n)\colon n \in \mathbb{N}\}$ is a countable local topological basis at $x$. Moreover, metric spaces (and metrizable spaces) are first countable, with a countable local topological basis at $x$ given by $\mathcal{L} = \{B(x,\epsilon_n)\colon n \in \mathbb{N}\}$.

We claim that the pointwise convergence topology is not first countable and is therefore not metrizable. Assume $\mathcal{L} = \{L_1, L_2, \dots\}$ is a countable local topological basis at the constant function 0. For each $L_i$, there is $B_i = B(0,\dots,0,t_{i1},\dots,t_{in_i},\epsilon_i)$ satisfying $0 \in B_i \subset L_i$. Since the set of points $t_{ij}$, $i = 1,2,\dots$, $1 \le j \le n_i$, is countable, there is $0 \le t_0 \le 1$ not equal to any of these $t_{ij}$. Now the constant function 0 is a point in the open subset $B(0,t_0,1)$. Therefore for the local topological basis $\mathcal{L}$, there is some $L_i$ satisfying $0 \subset L_i \subset B(0,t_0,1)$. This implies $B_i \subset B(0,t_0,1)$ for this particular $i$. But since $t_0$ is different from any of $t_{i1},\dots,t_{in_i}$, we can easily construct a function lying in $B_i$ but not in $B(0,t_0,1)$ (see Example 5.4.4 for detailed construction). The contradiction shows that the pointwise convergence topology cannot be first countable.

**Example 5.2.9.** A topological space $X$ is *separable* if it has a countable dense subset. The property is also a topological property.

The usual topology and the lower limit topology on $\mathbb{R}$ are separable, because the collection of rational numbers is dense. The $L_\infty$-topology is also separable, by taking the dense subset of polynomials with rational coefficients. The countable complement topology (see Exercises 4.1.1 and 4.3.3) on an uncountable set is not separable.

If a topology is separable, then coarser topologies are also separable. In particular, the $L_1$- and the pointwise convergence topologies are separable.

Exercise 5.2.10. Study the homeomorphisms between the topologies in Example 4.1.3.

1. Show that the following is a topological property: The complement of any open subset, except $\emptyset$, is finite. Then use the property to prove that the finite complement topology is not homeomorphic to any of the other topologies in the example.
2. Show that the Fréchet property in Exercise 4.5.22 is a topological property. Then use the property to prove that $\mathbb{R}_{(\leftarrow}$ is not homeomorphic to $\mathbb{R}_{(\leftrightarrow)}$, $\mathbb{R}_{[\leftrightarrow)}$, etc.
3. What topological property can be used to show that the lower limit topology is not homeomorphic to the topology induced by $\mathcal{B}_8$?

Exercise 5.2.11. Prove that the second countability implies the first countability and the separability. The lower limit topology is a counterexample to the converse.

Exercise 5.2.12. Prove that separable metric spaces must be second countable. Then use the discussions in Examples 5.2.7 and 5.2.8 to show that the lower limit topology is not metrizable.

## 5.3 Subspace

**Definition 5.3.1.** Let $X$ be a space with topology $\mathcal{T}$. The *subspace topology* on a subset $Y \subset X$ is

$$\mathcal{T}_Y = \{Y \cap U : U \text{ is an open subset of } X\}.$$

It is easy to verify that $\mathcal{T}_Y$ indeed satisfies the three axioms for a topology. For a subset $V$ of $Y$, the openness of $V$ as a subset of $Y$ may be different from the openness as a subset of $X$, as seen in Examples 2.3.2 and 2.3.3. The following describes the subspace topology in terms of topological (sub)bases.

**Lemma 5.3.2.** *If $\mathcal{B}$ is a topological basis for $X$, then*

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

*is a topological basis for the subspace topology on $Y$. Similar statement is also true for topological subbasis.*

*Proof.* We verify the two conditions in Lemma 4.4.3. Since $\mathcal{B} \subset \mathcal{T}$, we have $Y \cap B \in \mathcal{T}_Y$, which verifies the first condition. On the other hand, any $V \in \mathcal{T}_Y$ is of the form $V = Y \cap U$, with $U$ open in $X$. If $y \in V = Y \cap U$, then $y \in U$. Since $U$ is open in $X$, we have $y \in B \subset U$ for some $B \in \mathcal{B}$. Then $y \in Y \cap B \subset Y \cap U$ for some $Y \cap B \in \mathcal{B}_Y$. This verifies the second condition.

The claim about the subbasis is left as an exercise. □

The subspace topology can also be characterized by the natural inclusion map.

**Lemma 5.3.3.** *Let $Y$ be a subset of a topological space $X$. Then the subspace topology on $Y$ is the coarsest topology such that the inclusion map $i(y) = y\colon Y \to X$ is continuous. Moreover, a map $f\colon Z \to Y$ from a topological space $Z$ is continuous if and only if the map $i \circ f\colon Z \to X$ is continuous.*

Note that the map $i \circ f$ is simply $f$ viewed as a map into $X$.

*Proof.* The preimage of any subset $U \subset X$ under the inclusion map is $i^{-1}(U) = Y \cap U$. Therefore the continuity of $i$ means exactly that $Y \cap U$ is open in $Y$ for any open subset $U$ in $X$. Since the subspace topology on $Y$ consists of exactly such subsets, it is the coarsest topology with such property.

The continuity of $i \circ f$ means $(i \circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(Y \cap U)$ is open in $Z$ for any open subset $U$ in $X$. The meaning is the same as the continuity of $f$. □

A one-to-one map $f\colon Y \to X$ is an *embedding* if the induced invertible map $\phi\colon Y \to f(Y)$ is a homeomorphism, where $f(Y)$ has the subspace topology. Note that $\phi$ is invertible because it is one-to-one (by $f$ being one-to-one) and onto (by the definition of $\phi$). Moreover, $\phi$ is continuous because $f$ is continuous. Therefore the key for $f$ being an embedding is the continuity of $\phi^{-1}$.

**Example 5.3.1.** The usual topology on $\mathbb{R}$ is given by open intervals. By taking intersections with $[0,1]$, we get the topological basis for the subspace topology on $[0,1]$

$$\{(a,b)\colon 0 \le a < b \le 1\} \cup \{[0,b)\colon 0 < b \le 1\} \cup \{(a,1]\colon 0 \le a < 1\} \cup \{[0,1]\}.$$

**Example 5.3.2.** Let $L$ be a straight line in $\mathbb{R}^2_{\ell}$. The subspace topology on $L$ may be described by naturally identifying $L$ with the real line. If $L$ is not vertical, then the topological basis on $L$ is given by $\mathcal{B}_4$ in Example 4.1.3, which induces the lower limit topology. If $L$ is vertical, then the topological basis on $L$ is given by $\mathcal{B}_1$ in Example 4.1.3, which induces the usual topology.

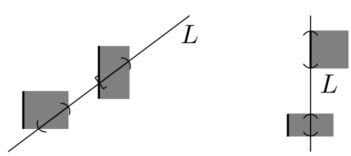



**Figure 5.3.1.** *subspace topology on a straight line in* $\mathbb{R}^2$

**Example 5.3.3.** Let $Y \subset C[0,1]$ be polynomials of degree $\leq 2$, with the subspace topology induced by the pointwise convergence topology. We claim that the natural map $p(a,b,c) = a + bt + ct^2 \colon \mathbb{R}^3_{\text{usual}} \to Y$ is a homeomorphism. In other words, the map $P(a,b,c) = a + bt + ct^2 \colon \mathbb{R}^3_{\text{usual}} \to C[0,1]_{\text{pt conv}}$ is an embedding.

First we show that $p$ is continuous. By Lemma 5.3.3, this is the equivalent to the continuity of $P$. The preimage of the topological subbasis of the pointwise convergence topology is

$$P^{-1}(B(\alpha, t_0, \epsilon)) = \{(a,b,c) \colon |\pi(a,b,c) - \alpha| < \epsilon\} = \pi^{-1}(\alpha - \epsilon, \alpha + \epsilon),$$

where (for fixed $t_0$)

$$\pi(a,b,c) = a + bt_0 + ct_0^2 \colon \mathbb{R}^3_{\text{usual}} \to \mathbb{R}_{\text{usual}}.$$

Since $\pi$ is continuous, the preimage $\pi^{-1}(\alpha - \epsilon, \alpha + \epsilon)$ is an open subset of $\mathbb{R}^3_{\text{usual}}$.

Next we construct a continuous inverse of $p$. The evaluations at three points produce a map $E(f) = \left(f(0), f\left(\frac{1}{2}\right), f(1)\right) \colon C[0,1]_{\text{pt conv}} \to \mathbb{R}^3_{\text{usual}}$. By the proof of Example 5.1.5, each evaluation is continuous. Then by Example 2.4.3, their combination is also continuous.

The composition

$$T(a,b,c) = E \circ P(a,b,c) = E \circ i \circ p(a,b,c) = \left(a, a + \frac{1}{2}b + \frac{1}{4}c, a + b + c\right)$$

is an invertible linear tranformation. Thus the composition $T^{-1} \circ E \circ i$ is the inverse of $p$. The inverse is continuous because $E$ and $i$ are continuous, and linear transformations are continuous maps from $\mathbb{R}^3_{\text{usual}}$ to itself.

Exercise 5.3.1. Consider the topological space in Example 4.3.5. Which subspaces are homeomorphic to $\{1,2,3\}$ with the topology $\big\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\big\}$? Moreover, find the closure of $\{4\}$ in the subspace $\{1,2,4\}$.

Exercise 5.3.2. Find the subspace topology on $\mathbb{N}$ induced by the topological bases in Example 4.1.3.

Exercise 5.3.3. Which subsets of $\mathbb{R}_{[\ )}$ have discrete subspace topology? What about subsets of $\mathbb{R}$ with topology $\{(a,\infty) \colon a \in \mathbb{Z}\} \cup \{\emptyset, \mathbb{R}\}$?

Exercise 5.3.4. Find the subspace topology of a straight line in $\mathbb{R}^2_{\llcorner\!\urcorner}$.

Exercise 5.3.5. Let $[n,\infty]$ be all the integers $\geq n$. Let $[n,\infty]$ have the subspace topology induced from the first topological basis in Exercise 4.6.3. What is the condition on $m$ and $n$ such that the subspaces $[m,\infty]$ and $[n,\infty]$ are homeomorphic?

Exercise 5.3.6. Prove that if $C[0,1]$ has the $L_1$- or $L_\infty$-topology, then the map $p$ in Example 5.3.3 is still a homeomorphism. In fact, we also have the homeomorphism for polynomials of other degrees.

Exercise 5.3.7. Prove Lemma 5.3.2 for topological subbasis.

Exercise 5.3.8. Let $Y \subset X$ be an open subset. Prove that a subset of $Y$ is open in $Y$ if and only if it is open in $X$.

Exercise 5.3.9. Let $Y \subset X$ have the subspace topology. Prove that $C \subset Y$ is closed if and only if $C = Y \cap D$ for some closed $D \subset X$.

Exercise 5.3.10. Let $Y \subset X$ have the subspace topology. Let $A \subset Y$. By considering $A$ as a subset of $X$ or $Y$, we have the closures $\bar{A}_X$ or $\bar{A}_Y$. Prove that $\bar{A}_Y = Y \cap \bar{A}_X$. Is there any relation between the limit points?

Exercise 5.3.11. When we say a subset $A$ is dense in another subset $U$, we could either mean $U \subset \bar{A}$, or $A \cap U$ is a dense subset of the subspace $U$.

1. Prove that for open $U$, the two definitions are equivalent.
2. For a general subset $U$, explain that the two definitions may not be equivalent.
3. Prove that if $A$ is a dense subset of $X$ and $U$ is open, then $A \cap U$ is dense in $U$.
4. Prove that if $X = \cup_i X_i$ is a union of open subsets, then $A$ is a dense subsets of $X$ if and only if $A$ is dense in each $X_i$.

Exercise 5.3.12. Let $X$ be a topological space. Let $Y \subset X$ and let $Z$ be an open subset of $X - Y$. Prove that any subset of $Z$ that is open in $Y \cup Z$ must be open in $X$. What if open is replaced by closed?

Exercise 5.3.13. Let $Y \subset X$ have the subspace topology. Prove that if $f\colon X \to Z$ is continuous, then the restriction $f|_Y\colon Y \to Z$ is continuous.

Exercise 5.3.14. Let $X = \cup_i X_i$ be a union of open subsets. Prove that $f\colon X \to Y$ is continuous if and only if the restrictions $f|_{X_i}\colon X_i \to Y$ are continuous, where $X_i$ has the subspace topology. Is the similar statement true for closed subsets?

Exercise 5.3.15. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ be unions of open subsets.

1. Suppose $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$, $f_0\colon X_1 \cap X_2 \to Y_1 \cap Y_2$ are homeomorphisms, such that $f_1(x) = f_0(x) = f_2(x)$ for $x \in X_1 \cap X_2$. Prove that $X$ and $Y$ are homeomorphic.
2. Suppose $X_1$ is homeomorphic to $Y_1$, $X_2$ is homeomorphic to $Y_2$, and $X_1 \cap X_2$ is homeomorphic to $Y_1 \cap Y_2$. Can you conclude that $X$ is homeomoprhic to $Y$?
3. Is it necessary to assume all subsets to be open?

Exercise 5.3.16. Let $Y$ be a subset of a metric space $X$. Then the restriction $d_Y$ of the metric $d_X$ of $X$ to $Y$ makes $Y$ into a *metric subspace*. Prove that the topology on $Y$ induced by $d_Y$ is the same as the subspace topology induced by the topology on $X$. The result shows the compatibility between the concepts of metric subspace and subspace topology.

Exercise 5.3.17. Let $Y \subset X$ have the subspace topology. Let $A, B \in Y$ be disjoint open subsets in the subspace topology.

1. If $X$ is a metric space, prove that $A = Y \cap U$ and $B = Y \cap V$ for some disjoint open subsets $U, V \subset X$.

2. Construct a topology on $X = \{1, 2, 3\}$, such that the subspace $Y = \{1, 2\}$ does not have the property in the first part.
3. Show that the similar statement does not hold for closed subsets, even in metric spaces.

## 5.4 Product

Given topological spaces $X$ and $Y$, how can a topology be introduced on the product $X \times Y$? Naturally, we would like to have

$$U \subset X \text{ and } V \subset Y \text{ are open} \implies U \times V \subset X \times Y \text{ is open.}$$

However, the collection

$$\mathcal{B}_{X\times Y} = \{U \times V : U \subset X \text{ and } V \subset Y \text{ are open}\}$$

is not closed under union and is therefore not a topology. On the other hand, the collection is indeed a topological basis by Lemma 4.1.2.

**Definition 5.4.1.** The *product topology* on $X \times Y$ is the topology induced by the topological basis $\mathcal{B}_{X\times Y}$.

The following gives a practical way of computing the product topology.

**Lemma 5.4.2.** *If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are topological bases for $X$ and $Y$, then*

$$\mathcal{B}_X \times \mathcal{B}_Y = \{B_1 \times B_2 : B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$$

*is a topological basis for the product topology. Moreover, if $\mathcal{S}_X$ and $\mathcal{S}_Y$ are topological subbases for $X$ and $Y$, then*

$$\begin{aligned} \mathcal{S}_X \times \mathcal{S}_Y &= \{S_1 \times S_2 : S_1 \in \mathcal{S}_X, S_2 \in \mathcal{S}_Y\}, \\ \mathcal{S}_{X\times Y} &= \{S_1 \times Y : S_1 \in \mathcal{S}_X\} \cup \{X \times S_2 : S_2 \in \mathcal{S}_Y\} \end{aligned}$$

*are topological subbases for the product topology.*

The claim about topological basis can be proved by using the criterion in Lemma 4.4.3, similar to the proof of Lemma 5.3.2. The claim about the subbasis is left as an exercise.

The product topology can also be characterized by natural maps.

**Lemma 5.4.3.** *Let $X$ and $Y$ be topological spaces. Then the product topology is the coarsest topology such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous. Moreover, a map $h(x) = (f(x), g(x)) : Z \to X \times Y$ is continuous if and only if its components $f = \pi_X \circ h : Z \to X$ and $g = \pi_Y \circ h : Z \to Y$ are continuous.*

*Proof.* The continuity of the projections means that if $U \subset X$ and $V \subset Y$ are open, then $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$ are open in $X \times Y$. This is the same as $U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$ being open for any open $U$ and $V$. Therefore the product topology is the coarsest topology satisfying the requirement.

The continuity of $h$ means that $h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ is open in $Z$ for any open $U \subset X$ and $V \subset Y$. By taking $U = X$ or $V = Y$, we see that this is equivalent to $f^{-1}(U)$ and $g^{-1}(V)$ being open, which is the same as the continuity of $f$ and $g$. $\square$

**Example 5.4.1.** Let $X = \{1, 2\}$ have the topology $\mathcal{T}_2$ in Example 4.3.4. The product topology on $X \times X$ is illustrated in Figure 5.4.1.

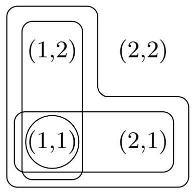



**Figure 5.4.1.** *product topology*

**Example 5.4.2.** The product topology $\mathbb{R}_{(\leftrightarrow)} \times \mathbb{R}_{(\leftrightarrow)}$ has a topological basis given by open rectangles. Therefore we have

$$\mathbb{R}_{(\leftrightarrow)} \times \mathbb{R}_{(\leftrightarrow)} = \mathbb{R}^2_{\square}.$$

Similarly, we have

$$\mathbb{R}_{[\leftrightarrow)} \times \mathbb{R}_{[\leftrightarrow)} = \mathbb{R}^2_{\llcorner\square}, \quad \mathbb{R}_{[\leftrightarrow)} \times \mathbb{R}_{(\leftrightarrow)} = \mathbb{R}^2_{[\square}.$$

**Example 5.4.3.** For any topological space $X$, the *diagonal map* $\Delta(x) = (x, x) \colon X \to X \times X$ is continuous by Lemma 5.4.3. As a matter of fact, the diagonal map is an embedding. In other words, the map $\delta(x) = (x, x) \colon X \to \Delta(X)$ is a homeomorphism, where the image $\Delta(X)$ has the subspace topology.

By Lemma 5.3.3 and the continuity of $\Delta$, we know $\delta$ is continuous. On the other hand, the inverse of $\delta$ is given by the composition $\Delta(X) \xrightarrow{i} X \times X \xrightarrow{\pi_1} X$, which is continuous because the inclusion $i$ and the projection $\pi_1$ are continuous.

**Example 5.4.4.** Consider the total evaluation map (as opposed to the evaluation at 0 only, in Example 5.1.5)

$$E(f, t) = f(t) \colon C[0, 1]_{\text{pt conv}} \times [0, 1]_{(\leftrightarrow)} \to \mathbb{R}_{(\leftrightarrow)}.$$

By Lemmas 5.1.3 and 5.4.2, the continuity of the map means the following: Given $f$, $t$, and $\epsilon > 0$, there is $B = B(a_1, \dots, a_n, t_1, \dots, t_n, \delta_1)$ and $\delta_2 > 0$, such that $f \in B$ and

$$g \in B \text{ and } |t' - t| < \delta_2 \implies |g(t') - f(t)| < \epsilon.$$

It turns out this never holds. For any $B$ and $\delta_2$, there is $t'$, such that

$$t' \neq t_1, \ \dots, \ t' \neq t_1; \ |t' - t| < \delta_2.$$

Then it is easy to construct a continuous function $g$ satisfying

$$g(t_1) = a_1,\ \ldots,\ g(t_n) = a_n;\ |g(t') - f(t)| > \epsilon.$$

This $g$ is a counterexample of the implication above. Therefore we conclude that $E$ is not continuous.

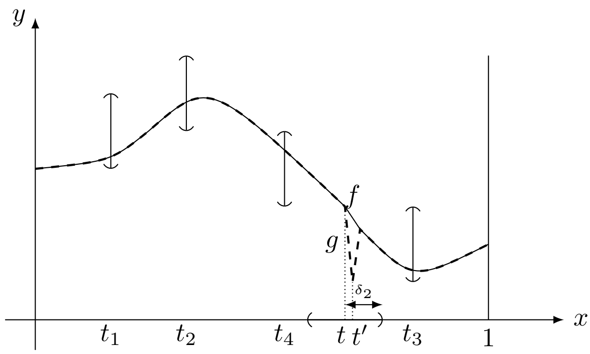



**Figure 5.4.2.** *Total evaluation is not continuous.*

Exercise 5.4.1. Find the closure of a straight line in $\mathbb{R}_{[\ )} \times \mathbb{R}_{(\ -}$.

Exercise 5.4.2. Find the closure of a solid triangle in $\mathbb{R}_{(\ )} \times \mathbb{R}_{\text{finite complement}}$.

Exercise 5.4.3. Study the continuity.

1. $\sigma(x, y) = x + y, \mu(x, y) = xy\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ has the topology induced by topological basis $\{(a, b)\colon a \in \mathbb{Z}\}$.
2. $\delta(x, y) = x - y\colon \mathbb{R}_{[\ )} \times \mathbb{R}_{(\ )} \to \mathbb{R}_{\text{finite complement}}$.
3. $\sigma(x, y) = x + y, \mu(x, y) = xy\colon \mathbb{R}_{[\ )} \times \mathbb{R}_{[\ )} \to \mathbb{R}_{[\ )}$.
4. $\sigma(x, y) = x + y, \mu(x, y) = xy\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ has the topology induced by the topological subbasis in Exercise 4.1.9.
5. $E(f, t) = f(t)\colon C[0, 1]_{\text{pt conv}} \times [0, 1]_{[\ )} \to \mathbb{R}_{[\ )}$.
6. $E(f, t) = f(t)\colon C[0, 1]_{\text{pt conv}} \times [0, 1]_{\text{finite complement}} \to \mathbb{R}_{\text{finite complement}}$.
7. $E(f, t) = f(t)\colon C[0, 1]_{L_\infty} \times [0, 1]_{[\ )} \to \mathbb{R}_{[\ )}$.
8. $\sigma(f, g) = f + g, \mu(f, g) = fg\colon C[0, 1] \times C[0, 1] \to C[0, 1]$, where $C[0, 1]$ has pointwise convergence topology.

Exercise 5.4.4. Prove that $\phi(f, g) = f(0) - g(0)\colon C[0, 1]_{\text{pt conv}} \times C[0, 1]_{\text{pt conv}} \to \mathbb{R}_{(\ )}$ is continuous. Then use this to conclude that $\{(f, g)\colon f(0) \geq g(0)\}$ is a closed subset of $C[0, 1]_{\text{pt conv}} \times C[0, 1]_{\text{pt conv}}$. Moreover, study the continuity of $\phi$ with respect to the other topologies on $C[0, 1]$.

Exercise 5.4.5. The map $\phi(f) = \left(f\left(\frac{t}{2}\right), f\left(\frac{t+1}{2}\right)\right) : C[0,1] \to C[0,1] \times C[0,1]$ "splits" a continuous function into two continuous functions, by restricting to $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. Prove that if $C[0,1]$ has the pointwise convergence topology, then $\phi$ is an embedding.

Exercise 5.4.6. Prove that if $f, g\colon X \to \mathbb{R}_{\leftrightarrow}$ are continuous, then for any constants $\alpha$ and $\beta$, the linear combination $\alpha f + \beta g$ and the product $fg$ are also continuous. Then use this to show that if $f, g, h\colon X \to \mathbb{R}_{\leftrightarrow}$ are continuous, then $\{x\colon f(x) < g(x) < h(x)\}$ is an open subset of $X$, and $\{x\colon f(x) \le g(x) \le h(x)\}$ is a closed subset.

Exercise 5.4.7. Prove Lemma 5.4.2 for topological subbasis.

Exercise 5.4.8. Describe a homeomorphism between $[0,1) \times [0,1)$ and $[0,1] \times [0,1)$, both having the usual topology. This shows that if $X \times Z$ is homeomorphic to $Y \times Z$, it does not necessarily follow that $X$ is homeomorphic to $Y$.

Exercise 5.4.9. Let $X$ and $Y$ be topological spaces and fix $b \in Y$. Prove that the natural map $i(x) = (x, b)\colon X \to X \times Y$ is an embedding of $X$ into $X \times Y$.

Exercise 5.4.10. Prove that the projection $\pi_X\colon X \times Y \to X$ maps open subsets to open subsets, but does not necessarily map closed subsets to closed subsets.

Exercise 5.4.11. Suppose $X$ and $Y$ are not empty. Prove that $X \times Y$ is discrete if and only if $X$ and $Y$ are discrete. What about the trivial topology or the finite complement topology?

Exercise 5.4.12. For maps $f\colon X \to Z$ and $g\colon Y \to W$, prove that $(f(x), g(y))\colon X \times Y \to Z \times W$ is continuous if and only if $f$ and $g$ are continuous.

Exercise 5.4.13. Prove that a map $f\colon X \to Y$ is continuous if and only if the *graph map* $\gamma(x) = (x, f(x))\colon X \to X \times Y$ is continuous. Moreover, prove that the graph map is actually an embedding.

Exercise 5.4.14. Prove properties of the product, assuming $A \subset X$ and $B \subset Y$ are not empty.

1. $A \times B$ is open $\iff$ $A$ and $B$ are open.
2. $A \times B$ is closed $\iff$ $A$ and $B$ are closed.
3. $\overline{A \times B} = \bar{A} \times \bar{B}$.
4. $(A \times B)' = A' \times \bar{B} \cup \bar{A} \times B'$.
5. $\partial(A \times B) = \partial A \times \bar{B} \cup \bar{A} \times \partial B$.

Exercise 5.4.15. Let $X$ and $Y$ be metric spaces. Then the product metric space $X \times Y$ may be defined by using the metric $d_{X \times Y}$ in Exercise 2.3.13. In fact, the exercise contains some other choices of the product metric, which induce the same topology on $X \times Y$. Prove that $d_{X \times Y}$ induces the product topology. The result shows the compatibility between the concepts of product metric and product topology.

## 5.5 Quotient

The sphere is obtained by collapsing the boundary of a disk to a point. Gluing the two opposing edges of a rectangle produces either a cylinder or a Möbius band. Further gluing the two ends of the cylinder produces either a torus or a Klein bottle.

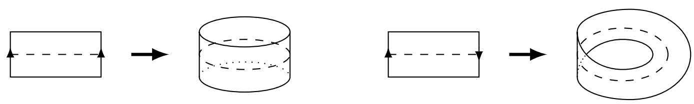


**Figure 5.5.1.** *glue two ends of the rectangle together in two ways*

**Figure 5.5.2.** *glue two ends of the cylinder together in two ways*

The topologies for the sphere, the cylinder, the Möbius band, etc, may be constructed by making use of the usual topologies on the disks and the rectangles. In general, given an *onto* map $f\colon X \to Y$ and a topology on $X$, a topology on $Y$ may be constructed. Naturally, the topology on $Y$ should make $f$ continuous. This means that if $U \subset Y$ were open, then $f^{-1}(U) \subset X$ should also be open. This leads to the following definition.

**Definition 5.5.1.** Given a topological space $X$ and an onto map $f\colon X \to Y$, the *quotient topology* on $Y$ is

$$\mathcal{T}_Y = \{U \subset Y : f^{-1}(U) \text{ is open in } X\}.$$

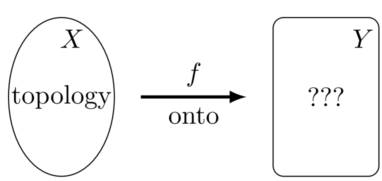



**Figure 5.5.3.** *quotient topology*

Strictly speaking, it needs to be verified that $\mathcal{T}_Y$ satisfies the three conditions for topology. This follows easily from $f^{-1}(\cup U_i) = \cup f^{-1}(U_i)$ and $f^{-1}(\cap U_i) = \cap f^{-1}(U_i)$.

Figure 5.5.4 shows some open subsets in the quotient topologies on the sphere and the cylinder. Figure 5.5.5 shows some open subsets in the quotient topology on the torus. They are consistent with the intuition.

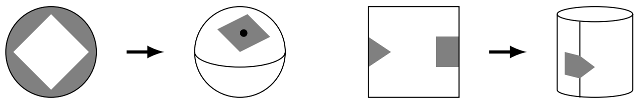


**Figure 5.5.4.** *quotient topologies on the sphere and the cylinder*

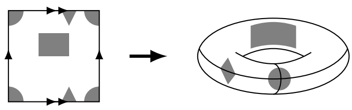


**Figure 5.5.5.** *quotient topology on the torus*

**Example 5.5.1.** Consider the map

$$f(x) = \begin{cases} -, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ +, & \text{if } x > 0 \end{cases} : \mathbb{R} \to \{+, 0, -\}.$$

If $\mathbb{R}$ has the usual topology, then the quotient topology is given in Figure 5.5.6.

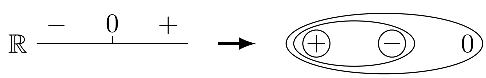



**Figure 5.5.6.** *a quotient topology on three points*

**Example 5.5.2.** Let $\mathbb{R}' = (\mathbb{R} - \{0\}) \cup \{0_+, 0_-\}$ be the *line with two origins*. An obvious onto map $f\colon \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R}'$ sends the origins of the two copies of $\mathbb{R}$ respectively to $0_+$ and $0_-$. Let $\mathbb{R} \sqcup \mathbb{R}$ have the obvious usual topology: The open subsets are of the form $U \sqcup V$, with $U$ and $V$ being open subsets of respective copies of $\mathbb{R}_{(-)}$.

What are the open subsets $U \subset \mathbb{R}'_{\text{quotient}}$? Write $U = V \cup O$, with $V \subset \mathbb{R} - \{0\}$ and $O \subset \{0_+, 0_-\}$. If $O = \emptyset$ (so that $U = V$), then $f^{-1}(U) = V \sqcup V$. Therefore $U$ is open if and only if $V$ is open. If $O = \{0_+\}$, then $f^{-1}(U) = (V \cup \{0\}) \sqcup V$. Therefore $U$ is open if and only if $V \cup \{0\}$ is open. Similar argument can be made for the cases $O = \{0_-\}$ or $O = \{0_+, 0_-\}$. Then it is easy to conclude that the open subsets in $\mathbb{R}'_{\text{quotient}}$ are either open subsets of $\mathbb{R} - \{0\}$, or open subsets of $\mathbb{R}$ that include at least one (or both) of $0_+$ and $0_-$.

Exercise 5.5.1. What are the quotient topologies of the discrete and the trivial topologies?

Exercise 5.5.2. In Example 5.5.1, what is the quotient topology on $\{+, 0, -\}$ if $\mathbb{R}$ has the topology induced by a topological basis in Example 4.1.3?

Exercise 5.5.3. Let $X$ and $Y$ be topological spaces and let $f\colon X \to Y$ be an onto map that takes open subsets to open subsets.

**Figure 5.5.7.** *open subsets in real line with two origins*

1. Prove that the topology on $Y$ is finer than the quotient topology.
2. Prove that if $f$ is continuous, then the topology on $Y$ is the quotient topology.

In particular, by Exercise 5.4.10, for nonempty topological spaces $X$ and $Y$, the quotient topology on $X$ induced by the projection $\pi_X \colon X \times Y \to X$ is the original topology on $X$.

Exercise 5.5.4. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on $X$, such that $\mathcal{T}$ is coarser than $\mathcal{T}'$. Let $Y$ be any set of more than one element and fix $b \in Y$. Verify that

$$\{(U \times Y) \cup (V \times b) \colon U \in \mathcal{T}, V \in \mathcal{T}'\}$$

is a topology on $X \times Y$. Moreover, show that the quotient topology on $X$ induced by the projection $X \times Y \to X$ is $\mathcal{T}$. The example shows that a quotient map may not take open subsets to open subsets.

Exercise 5.5.5. Let $X$ be a topological space and let $f \colon X \to Y$ be an onto map.

1. Prove that $C \subset Y$ is closed in the quotient topology if and only if $f^{-1}(C)$ is closed in $X$.
2. Prove that if $Y$ is a topological space, such that $f$ takes closed subsets to closed subsets, then the topology on $Y$ is finer than the quotient topology.
3. Prove that if $Y$ is a topological space, such that $f$ is continuous and takes closed subsets to closed subsets, then the topology on $Y$ is the quotient topology.
4. Show that in general, the quotient map may not take closed subsets to closed subsets.

Let $f \colon X \to Y$ be an onto map. Let $\mathcal{B}$ be a topological basis of $X$. Then the openness of a subset $U \subset Y$ in the quotient topology means openness of $f^{-1}(U)$ with respect to $\mathcal{B}$. In other words, if $f(x) \in U$, then $x \in B$ and $f(B) \subset U$ for some $B \in \mathcal{B}$.

The quotient topology can also be characterized by the natural map.

**Lemma 5.5.2.** *Let $f \colon X \to Y$ be an onto map from a topological space $X$. Then the quotient topology on $Y$ is the finest topology such that the quotient map $f$ is continuous. Moreover, a map $g \colon Y \to Z$ to a topological space $Z$ is continuous if and only if $g \circ f \colon X \to Z$ is continuous.*

*Proof.* Suppose $f$ is continuous with respect to a topology $\mathcal{T}$ on $Y$. Then for any $U \in \mathcal{T}$, the preimage $f^{-1}(U)$ is open in $X$. Therefore $U$ is also open in the quotient topology. This shows that $\mathcal{T}_Y$ is the finest topology on $Y$ such that $f$ is continuous.

The continuity of $g \circ f$ means that for any open $U \subset Z$, the preimage $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in $X$. By the definition of the quotient topology, the openness of $f^{-1}(g^{-1}(U))$ in $X$ means exactly that $g^{-1}(U)$ is open in the quotient topology on $Y$. Therefore the continuity of $g \circ f$ means that $g^{-1}(U)$ is open in $Y$ for any open $U \subset Z$, which is the continuity of $g$. $\square$

**Example 5.5.3.** The map $\phi(t) = e^{2\pi i t} \colon \mathbb{R}_{(\,)} \to S^1$ is onto the unit circle. The openness of a subset $U \subset S^1$ in the quotient topology means that for any $\phi(t) \in U$, there is $B = (t - \delta, t + \delta)$, such that $\phi(B) \subset U$. Note that $\phi(B)$ is an open arc on the unit circle around $\phi(t)$, and such open arcs form a topological basis of $S^1$ as a topological subspace of $\mathbb{R}^2$. Therefore the quotient topology is the same as the usual subspace topology on the circle.

Lemma 5.5.2 further tells us that continuous periodic functions of period $2\pi$ (see Example 1.4.6) are in one-to-one correspondence with the continuous functions on the usual circle.

**Example 5.5.4.** What is the quotient topology induced by the onto map $f(x) = x^2 \colon \mathbb{R}_{[\,)} \to [0, \infty)$? A subset $U \subset [0, \infty)$ is open in the quotient topology if and only if the preimage $f^{-1}(U)$ is open in the lower limit topology. This means

$$f(x) \in U \implies f[x, x + \epsilon) \subset U \text{ for some } \epsilon > 0.$$

Since any $a \in U - \{0\}$ has a positive square root $x = \sqrt{a}$ and a negative square root $-x$, for $0 < \epsilon < x$, the statement above becomes

$$x^2 = a \in U \implies [a, (x + \epsilon)^2) \text{ and } ((-x + \epsilon)^2, a] \subset U \text{ for some } x > \epsilon > 0.$$

The right side means $(a - \epsilon_1, a + \epsilon_2) \subset U$, where $\epsilon_1 = \epsilon(2x - \epsilon)$ and $\epsilon_2 = \epsilon(2x + \epsilon)$. Similarly, if $0 \in U$, then the statement becomes

$$0 \in U \implies [0, \epsilon^2) \subset U \text{ for some } \epsilon > 0.$$

Thus we conclude that the quotient topology is the usual topology.

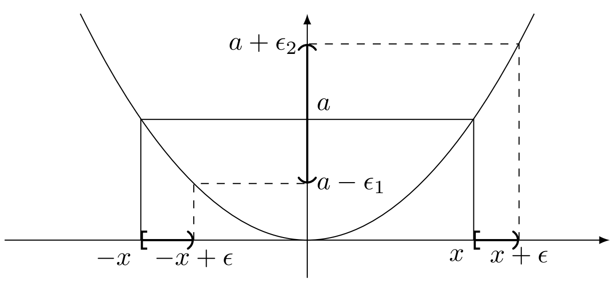



**Figure 5.5.8.** *quotient topology induced by $x^2$*

**Example 5.5.5.** What is the quotient topology induced by the addition map $\sigma(x, y) = x + y \colon \mathbb{R}_{[\,)} \times \mathbb{R}_{[\,)} \to \mathbb{R}$?

By Example 5.1.4, the map $\sigma$ is continuous if $\mathbb{R}$ has the lower limit topology. By Lemma 5.5.2, therefore, the quotient topology is finer than the lower limit topology. On the other hand, the $x$-axis map $i(x) = (x, 0)\colon \mathbb{R}_{[\to)} \to \mathbb{R}_{[\to)} \times \mathbb{R}_{[\to)}$ is continuous. Therefore $\sigma \circ i(x) = x\colon \mathbb{R}_{[\to)} \to \mathbb{R}_{\text{quotient}}$ is also continuous. This implies that the quotient topology is coarser than the lower limit topology. Therefore we conclude that the quotient topology is the lower limit topology.

Exercise 5.5.6. Compute the quotient topology.

1. $\phi(t) = e^{2\pi i t}\colon [0, 1]_{(\leftrightarrow)} \to S^1$.
2. $\delta(x, y) = x - y\colon \mathbb{R}_{[\to)} \times \mathbb{R}_{[\to)} \to \mathbb{R}$.
3. $\mu(x, y) = xy\colon \mathbb{R}_{[\to)} \times \mathbb{R}_{[\to)} \to \mathbb{R}$.
4. $\sigma(x, y) = x + y\colon \mathbb{R}_{(\leftarrow} \times \mathbb{R}_{(\leftarrow} \to \mathbb{R}$.
5. $\sigma(x, y) = x + y\colon \{1, 2\}^2 \to \{2, 3, 4\}$, where $\{1, 2\}$ has the topology $\big\{\emptyset, \{1\}, \{1, 2\}\big\}$.
6. $f(x) = x^2\colon \mathbb{R}^2_{\square} \to [0, \infty)$.
7. $E(f) = \left(f(0), f\left(\frac{1}{2}\right), f(1)\right) : C[0, 1]_{\text{pt conv}} \to \mathbb{R}^3$.

Exercise 5.5.7. Let $f$ be a continuous function (in the usual calculus sense) on a closed and bounded interval $[a, b]$. Let $\alpha = \min_{[a,b]} f$ and $\beta = \max_{[a,b]} f$. Then $f$ may be considered as an onto map $[a, b] \to [\alpha, \beta]$.

1. Prove that if $[a, b]$ has the usual topology, then the quotient topology on $[\alpha, \beta]$ is the usual topology.
2. Prove that if $f$ is increasing and $[a, b]$ has the lower limit topology, then the quotient topology on $[\alpha, \beta]$ is the lower limit topology.
3. Find a general sufficient condition on $f$ so that if $[a, b]$ has the lower limit topology, then the quotient topology on $[\alpha, \beta]$ is the usual topology.

Exercise 5.5.8. Let $f\colon X \to Y$ be an onto map. Let $\mathcal{B}$ be a topological basis of $X$. Is $f(\mathcal{B}) = \{f(B)\colon B \in \mathcal{B}\}$ a topological basis of the quotient topology?

Exercise 5.5.9. Let $X$ and $Y$ be topological spaces. Let $f\colon X \to Y$ be an onto continuous map with the following property: A map $g\colon Y \to Z$ to a topological space $Z$ is continuous if and only if $g \circ f\colon X \to Z$ is continuous. Is it necessarily true that $Y$ has the quotient topology?

Exercise 5.5.10. Let $X$ be a topological space. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be onto maps. Then $f$ induces a quotient topology on $Y$, and with this quotient topology, $g$ further induces a quotient topology on $Z$. On the other hand, the composition $g \circ f\colon X \to Z$ induces a quotient topology on $Z$. Prove that the two topologies on $Z$ are the same.

Exercise 5.5.11. Let $X$ be a topological space. Let $Y \subset X$ be a subset. Let $f\colon X \to Z$ be a map such that the restriction $f|_Y\colon Y \to Z$ is onto. Do $f$ and $f|_Y$ induce the same quotient topology on $Z$?

Exercise 5.5.12. Let $f : X \to Y$ be an onto map. Then for any topology $\mathcal{T}_Y$ on $Y$, we have the *pullback topology* (see Exercise 4.3.8)

$$f^*\mathcal{T}_Y = \{f^{-1}(V) : V \in \mathcal{T}_Y\}.$$

On the other hand, for any topology $\mathcal{T}_X$ on $X$, we denote the quotient topology (which may be called the *pushforward topology*) on $Y$ by $f_*\mathcal{T}_X$.

1. Given a topology on $Y$, prove that the pullback topology is the coarsest topology on $X$ such that $f$ is continuous.
2. Prove that for any topology $\mathcal{T}_Y$ on $Y$, we always have $f_*(f^*\mathcal{T}_Y) = \mathcal{T}_Y$.
3. Is it true that $f^*(f_*\mathcal{T}_X) = \mathcal{T}_X$ for any topology $\mathcal{T}_X$ on $X$?

An important application of the quotient topology is the topology on the space obtained by attaching one space to another. Specifically, let $X$ and $Y$ be topological spaces. Let $A \subset Y$ be a subset and let $f\colon A \to X$ be a continuous map. Construct $X \cup_f Y$ by starting with the disjoint union $X \sqcup Y$ and then identifying $a \in A \subset X$ with $f(a) \in Y$. The process is the *attaching* (or *gluing*) of $Y$ to $X$ along $A$.

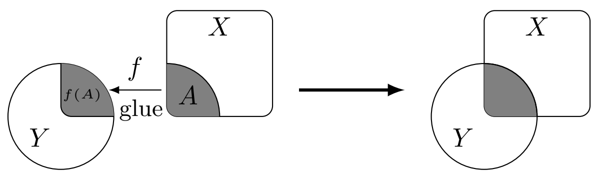



**Figure 5.5.9.** *glue $Y$ to $X$ along $A$*

For example, let $X$ and $Y$ be two copies of the disk $B^n$. Let $A = S^{n-1} \subset Y$ be the boundary sphere. Let $f(a) = a\colon A \to X$ be the inclusion of the boundary sphere into $X$. Then $X \cup_f Y$ is the sphere $S^n$. For another example, let $X$ be a single point and $Y = B^n$. Let $A = S^{n-1} \subset Y$ be the boundary sphere. Let $f(a) = a\colon A \to X$ be constant map to the point. Then $X \cup_f Y$ is again the sphere $S^n$.

In the most strict set-theoretical language, the set $X \cup_f Y$ is the quotient set of $X \sqcup Y$, subject to the relation $a \sim f(a)$ for any $a \in X$. This is not yet an equivalence relation because the relations $f(a) \sim a$ (for any $a \in A$), $x \sim x$ (for any $x \in X$), $y \sim y$ (for any $y \in Y$) must also hold. Moreover, if $a, b \in A$ satisfy $f(a) = f(b)$, then we should also have $a \sim b$. After adding all these additional relations, an equivalence relation on $X \sqcup Y$ is obtained, called the equivalence relation induced by $a \sim f(a)$ (see Exercise 1.4.4). Then $X \cup_f Y$ may be defined as the quotient set of this equivalence relation.

What is the suitable topology on $X \cup_f Y$? First we have a natural topology on $X \sqcup Y$ by taking open subsets to be $U \sqcup V$, where $U \subset X$ and $V \subset Y$ are open. Then the natural onto map $X \sqcup Y \to X \cup_f Y$ induces the quotient topology on $X \cup_f Y$. Indeed in the two examples above, the usual topologies on $X$ and $Y$ lead to the usual topology on $S^n$. However, such a choice may sometimes lead to unexpected consequences, as illustrated by the example below. For the applications in the subsequent sections, we do not need to worry too much because $X$ and $Y$ are often bounded and closed subsets of Euclidean spaces and the strange phenomenon will not happen.

**Example 5.5.6.** Let $Z$ be a topological space and $Z = X \cup Y$. Then $X$ and $Y$ have the subspace topologies. Think of $A = X \cap Y$ as a subset of $Y$ and let $i\colon A \to X$ be the inclusion. Then $X \cup_i Y = X \cup_A Y$ is easily identified with $Z$.

Let us compare the quotient topology and the original topology on $Z$. For the two to be the same means that $U$ is an open subset of $Z$ in the original topology if and only if $U \cap X$ is open in $X$ and $U \cap Y$ is open in $Y$. This is indeed the case when both $X$ and $Y$ are open subsets of $Z$. By replacing "open" with "closed" throughout the discussion, we also know that the quotient topology and the original topology are the same if both $X$ and $Y$ are closed subsets of $Z$.

In general, however, the claim is not true. Consider $Z = \mathbb{R}_{\leftrightarrow}$, $X = (-\infty, 0]$, $Y = (0, \infty)$, and $A = \emptyset$. Then $U$ is open in the quotient topology if and only if $U \cap (-\infty, 0]$ is open in $(-\infty, 0]_{\leftrightarrow}$, and $U \cap (0, \infty)$ is open in $(0, \infty)_{\leftrightarrow}$. In particular, $(-1, 0]$ is open in the quotient topology but not in the usual topology.

Exercise 5.5.13. Prove that in Example 5.5.6, the original topology on $Z$ is coarser than the quotient topology.

Exercise 5.5.14. Prove that a map $X \cup_f Y \to Z$ is continuous if and only if the induced maps $X \to Z$ and $Y \to Z$ are continuous.

## Chapter 6

# Complex

## 6.1 Simplicial Complex

The theory of point set topology (open, closed, continuity, connected, compact, etc.) lays down the rigorous mathematical foundation for problems of topological nature. However, to actually solve many topological problems, another type of theory that catches the internal structures is needed. The graph theory is one such example.

A very general strategy for solving problems is to decompose the objects into simple pieces. By studying the simple pieces and (more importantly) how the simple pieces combine to produce the object, the solutions to the problems may be found. For example, every integer can be decomposed into a product of prime numbers. In this way, the prime numbers are the simple pieces and the product is the method for putting simple pieces together. For another example, the world economy may be analysed by considering the economies of individual countries and then studying the trade between different countries.

In a simplicial complex, a topological space is decomposed into triangles at various dimensions, called simplices. In a CW-complex, a topological space is decomposed into balls, called cells. From such decompositions, the Euler number (and more importantly, the homology theory) may be defined and computed. Other important topological decompositions include handlebody and stratification, which we will not discuss in this course.

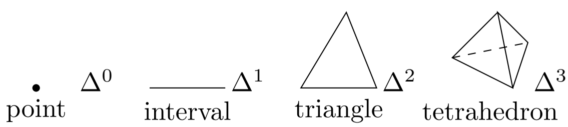



**Figure 6.1.1.** *simplices up to dimension* 3

A 0-dimensional simplex is a point. A 1-dimensional simplex is an interval. A 2-dimensional simplex is a triangle. A 3-dimensional simplex is a tetrahedron. In general, and $n$-dimensional simplex is the $n$-dimensional version of triangle. Specifically, define vectors $v_0, v_1, \dots, v_n$ in a Euclidean space to be *affinely independent* if

$$\begin{matrix} c_0v_0 + c_1v_1 + \cdots + c_nv_n = 0 \\ c_0 + c_1 + \cdots + c_n = 0 \end{matrix} \implies c_0 = c_1 = \cdots = c_n = 0.$$

This is equivalent to the *linear independence* of the vectors $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$. The convex hull spanned by the affinely independent vectors (called *vertices*) is an *$n$-simplex*

$$\sigma^n(v_0, v_1, \dots, v_n) = \{\lambda_0v_0 + \lambda_1v_1 + \cdots + \lambda_nv_n : 0 \le \lambda_i \le 1, \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1\}.$$

Figure 6.1.2 shows the simplices with affinely independent vectors $0, e_1, e_2, \dots, e_n$ in $\mathbb{R}^n$ as vertices.

A simplex contains many simplices of lower dimensions. For any selection $\{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}$ out of all the vertices $\{v_0, v_1, \dots, v_n\}$, the simplex $\sigma(v_{i_0}, v_{i_1}, \dots, v_{i_k})$ is called a *face* of the simplex $\sigma(v_0, v_1, \dots, v_n)$. Figure 6.1.3 shows all the faces of a 2-simplex.

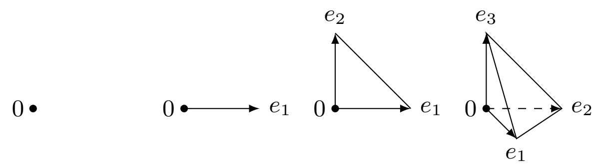



**Figure 6.1.2.** *simplices with* $0, e_1, e_2, \ldots,$ *as vertices*

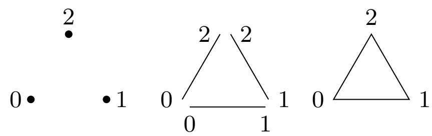



**Figure 6.1.3.** *seven faces of a 2-simplex*

Exercise 6.1.1. How many $k$-dimensional faces does an $n$-dimensional simplex have?

The following definition describes how simplices can be put together to form more complicated objects.

**Definition 6.1.1.** A *simplicial complex* is a collection $K$ of simplices in a Euclidean space $\mathbb{R}^N$, such that

- $\sigma \in K \implies$ faces of $\sigma \in K$.
- $\sigma, \tau \in K \implies \sigma \cap \tau$ is either empty or one common face of $\sigma$ and $\tau$.
- $K$ is locally finite: Every point of $\mathbb{R}^N$ has a neighborhood that intersects with only finitely many simplices in $K$.

Among the conditions, the second one requires the simplices to be arranged nicely and is the most important condition. Figure 6.1.4 shows some examples that do not satisfy the second condition. The condition also does not allow two different edges to meet at both ends. The first condition is not important because it can be met by simply adding all the faces of existing simplices to the collection. The third condition is not important, at least for this course, because all simplicial complexes studied in this course are finite.

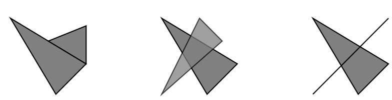


**Figure 6.1.4.** *counterexamples of the second condition*

**Definition 6.1.2.** The *geometrical realization* $|K|$ of a simplicial complex $K$ is the union of all simplices in $K$. A subset $X$ of $\mathbb{R}^N$ is a *polyhedron* if $X = |K|$ for some simplicial complex $K$, and we also say $K$ is a *triangulation* of $X$.

A simplex $\Delta^n$ and all its faces form a simplicial complex. The simplices in this simplicial complex are in one-to-one correspondence with nonempty subsets of $\{0, 1, \dots, n\}$. The geometrical realization of this simplicial complex is $\Delta^n$. Therefore $\Delta^n$ is a polyhedron. Figure 6.1.5 shows many more sophisticated examples of polyhedra and the simplicial complexes that triangulate them.

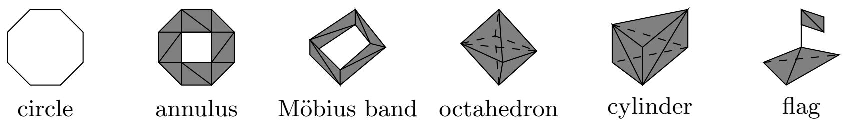



**Figure 6.1.5.** *examples of triangulation*

Figure 6.1.6 shows that a polyhedron may have different triangulations.

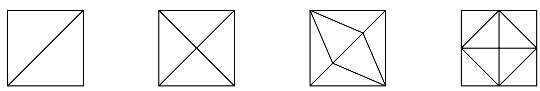


**Figure 6.1.6.** *four triangulations of the square*

Since the simplices are never "curved", polyhedra are "straight" objects. On the other hand, spaces such as circle, annulus, Möbius[10] band and torus are usually considered as curved. To study their topological properties by simplicial method, we need to "straighten" them by homeomorphisms to suitable polyhedra. Figure 6.1.7 is the straightening of the sphere.

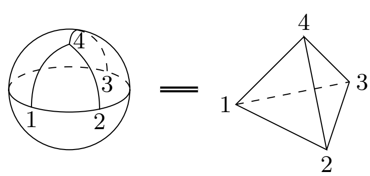



**Figure 6.1.7.** *straighten sphere*

In practice, many spaces are too complicated to draw in real shape (and even more difficult to draw in straightened shape). Thus simplicial complexes are often presented as a bunch of simplices with detailed instruction on how the simplices are glued together along their faces. We illustrate this through the example of torus.

[10] August Ferdinand Möbius, born November 17, 1790 in Schulpforta, Saxony (now Germany), died September 26, 1868 in Leipzig, Germany. Möbius discovered his band in 1858. Many mathematical concepts are named after him, including Möbius transform, Möbius function and Möbius inversion formula.

Note that the torus is obtained by identifying the opposite edges of a square as in Figure 6.1.8. Thus triangulations of the torus may be constructed by considering triangulations of the square.

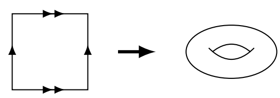


**Figure 6.1.8.** *construct torus from square*

Specifically, consider triangulations of the square in Figure 6.1.9. To determine which ones give trianglations of the torus, we need to examine the second condition in the definition of the simplicial complex, keeping in mind the identifications between the boundary points of the square. When the condition is applied to the case $\sigma$ is a vertex of $\tau$, we find that *vertices of any simplex are not identified.* When the condition is applied to general intersections $\sigma \cap \tau$, we find that *it is sufficient to consider the intersections of top dimensional simplices only.* Of course, strictly speaking, it needs to be proved that if the two key points are satisfied, then indeed these simplices can be put in a Euclidean space of sufficiently high dimension in a straight way. The fact is true and the detailed proof is omitted.

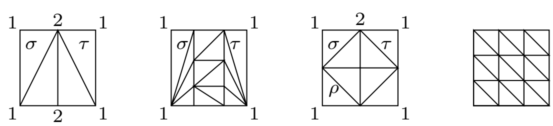



**Figure 6.1.9.** *triangulations of the torus?*

In Figure 6.1.9, the first triangulation fails both key points: Two vertices of $\sigma$ are identified, and the intersection $\sigma \cap \tau$ consists of two common faces (one edge and one vertex). The second is not a triangulation of the torus because two vertices of $\sigma$ are again identified. The third satisfies the first key point but fails the second one because the intersection $\sigma \cap \tau$ consists of one edge and one vertex. The similar failure happens for the intersection $\sigma \cap \rho$, which consists of two edges. Only the fourth one is a triangulation of the torus.

Exercise 6.1.2. In Figure 6.1.10, which are triangulations of the torus?

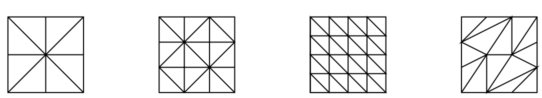


**Figure 6.1.10.** *triangulations of the torus?*

Exercise 6.1.3. The Klein[11] bottle is obtained by identifying the opposite edges of a square as in Figure 6.1.11. Find a triangulation of the square that gives a triangulation of the Klein bottle.

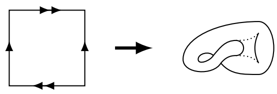


**Figure 6.1.11.** *construct Klein bottle from square*

Exercise 6.1.4. The (2-dimensional) real projective space $P^2$ may be obtained from a disk $B^2$ by identifying the opposite points of the boundary circle (see Figure 6.1.12). Find a triangulation of $P^2$.

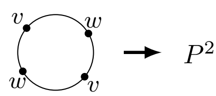



**Figure 6.1.12.** *real projective space*

## 6.2 CW-Complex

Simplices are the simplest pieces that can be used as the building blocks for topological spaces. However, simplicity comes at a cost: The decomposition often gets very complicated and even simple spaces require lots of simplices to build. Here is an analogy: The economy of an individual person is much simpler than the world economy. However, if individual persons are used as the units of study for the world economy, then the whole system would be so complicated that the study becomes impractical. On the other hand, if individual countries as used as the (more sophisticated) units of study, then the whole system is more manageable and easier to study.

The concept of CW-complexes strikes a good balance between simplicity and usability. The building blocks for CW-complexes are the closed balls $B^n$, called *cells*. A *CW-structure* on a space $X$ is a sequence of subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots \subset X = \cup X^n,$$

such that $X^n$ is obtained from $X^{n-1}$ by *attaching* (or *gluing*) some $n$-dimensional cells, and $X$ carries a suitable topology. The subspace $X^k$ is the *$k$-skeleton* of the CW-structure. If $X = X^n$, then $X$ is an *$n$-dimensional CW-complex.*

[11]Felix Christian Klein, born April 25, 1849 in Düsseldorf, Prussia (now Germany), died June 22, 1925 in Göttingen, Germany. Klein described his bottle in 1882. In 1872, Klein proposed the Erlangen program that classifies geometries by the underlying symmetry groups. The program profoundly influenced the evolution of mathematics.

A graph $G$ is a 1-dimensional CW-complex. $G^0$ is all the vertices, and $G = G^1$ is obtained by attaching the end points of edges to the vertices. The sphere $S^2$ is a 2-dimensional CW-complex, with a CW-structure given by one 0-cell and one 2-cell. The torus $T^2$ is a 2-dimensional CW-complex, with one 0-cell, two 1-cells, and one 2-cell.

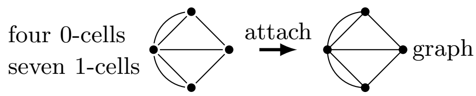



**Figure 6.2.1.** *CW-structure of a graph*

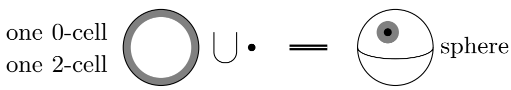



**Figure 6.2.2.** *CW-structure of sphere*

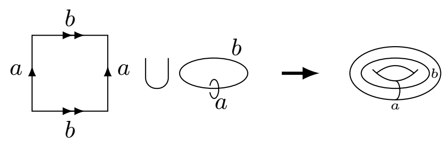



**Figure 6.2.3.** *CW-structure of torus*

Exercise 6.2.1. Find CW-structures for the Möbius band, the Klein Bottle and the projective space $P^2$.

To be more specific, the process of attaching a cell is illustrated in Figure 6.2.4. For a topological space $X$ and a continuous map $f\colon S^{n-1} \to X$, construct a set

$$X \cup_f B^n = X \sqcup \mathring{B}^n,$$

where $\mathring{B}^n = B^n - S^{n-1}$ is the interior of the unit ball. There is an onto map

$$F(a) = \begin{cases} a, & \text{if } a \in X \\ f(a), & \text{if } a \in S^{n-1} \\ a, & \text{if } a \in \mathring{B}^n \end{cases} : X \sqcup B^n \to X \cup_f B^n.$$

On the disjoint union $X \sqcup B^n$ is the natural topology

$$\{U \cup V : U \subset X \text{ and } V \subset B^n \text{ are open}\}.$$

Then the onto map $F$ induces the quotient topology on $X \cup_f B^n$.

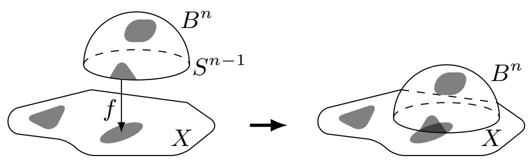



**Figure 6.2.4.** *open subsets after gluing a disk*

More generally, given several continuous maps $f_i\colon S^{n-1} \to X$, $i = 1, \dots, k$, we can similarly construct a topological space $X \cup_{\{f_i\}} (\sqcup_{i=1}^k B^n)$ with the suitable quotient topology. The space is said to be obtained by attaching $k$ $n$-cells to $X$ along the *attaching maps* $f_1, \dots, f_k$.

Thus a *finite CW-complex* is constructed by the following steps. Start with finitely many discrete points $X^0$ as the 0-skeleton. Next attach finitely many intervals to $X^0$ along the ends of the intervals to get a graph $X^1$, which is the 1-skeleton. Then attach finitely many 2-disks to $X^1$ along the maps from the boundary circles of the 2-disks to $X^1$. The result is the 2-skeleton $X^2$. Keep attaching finitely many disks of higher dimensions and each time equip the new skeleton with the quotient topology. The finite CW-complex, together with the quotient topology, is constructed after finitely many steps.

The process also works for infinite CW-complexes. However, some local finiteness condition is needed in describing the topology. Finally, note that with simplices as cells, any triangulation is a CW-structure.

## 6.3 Projective Space

The $n$-dimensional *real projective space* is

$$\begin{aligned} P^n &= \{\text{all straight lines in } \mathbb{R}^{n+1} \text{ passing through the origin}\} \\ &= \{\text{all 1-dimensional linear subspaces of } \mathbb{R}^{n+1}\}. \end{aligned}$$

Since there is only one straight line in $\mathbb{R}^1$, the 0-dimensional real projective space $P^0$ is a single point. Also note that the straight lines in $\mathbb{R}^2$ passing through the origin are in one-to-one correspondence with points in the upper half of the unit circle, with two ends of the upper half identified (both representing the $x$-axis). Consequently, the 1-dimensional real projective space $P^1$ is a circle.

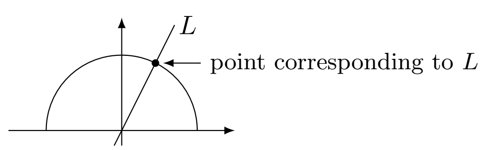



**Figure 6.3.1.** *Why is $P^1$ a circle?*

There are several useful ways of presenting the real projective space in general. Since a straight line passing through the origin is exactly a 1-dimensional linear subspace, which is the span of a *nonzero* vector, we have an onto map

$$f(v) = [v] = \text{span}\{v\} \colon \mathbb{R}^{n+1} - \{0\} \to P^n.$$

Since two nonzero vectors $u$ and $v$ span the same line if and only if $u = \lambda v$ for some $\lambda \in \mathbb{R} - \{0\}$, the projective space is the quotient set

$$P^n = \frac{\mathbb{R}^{n+1} - \{0\}}{\mathbb{R} - \{0\}} = \frac{\mathbb{R}^{n+1} - \{0\}}{v \sim \lambda v, \lambda \in \mathbb{R} - \{0\}}.$$

The usual topology on $\mathbb{R}^{n+1} - \{0\}$ induces the quotient topology on $P^n$.

Another way of presenting the projective space is based on the same idea, with the additional observation that using unit vectors is sufficient to generate all straight lines. Since the unit vectors in $\mathbb{R}^{n+1}$ form the unit sphere $S^n$, this leads to the onto map

$$\pi(v) = [v] = \text{span}\{v\} \colon S^n \to P^n.$$

Moreover, because $\pi(u) = \pi(v)$ is equivalent to $u = \pm v$, the real projective space is obtained by identifying the antipodal points on the unit sphere

$$P^n = \frac{S^n}{v \sim -v}.$$

Because $\mathbb{R}^{n+1} - \{0\}$ is homeomorphic to $S^n \times (0, \infty)$, the usual topology on $S^n$ and the usual topology on $\mathbb{R}^{n+1} - \{0\}$ induce the same quotient topology on $P^n$.

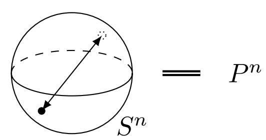



**Figure 6.3.2.** *The projective space is obtained by identifying antipodal points.*

Using the sphere model, it is easy to deduce from $S^0 = \{-1, 1\}$ that $P^0$ is a single point. Figure 6.3.3 shows that $P^1$ is homeomorphic to $S^1$, and the quotient map $S^1 \to P^1$ is "wrapping $S^1$ twice on itself". If we think of $S^1$ as consisting of complex numbers $z$ of norm 1, then the quotient map is $z \to z^2$.

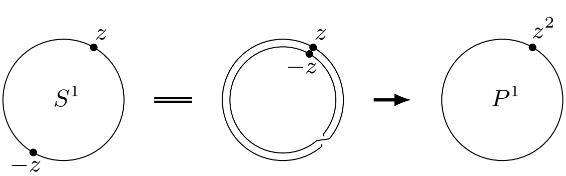



**Figure 6.3.3.** $S^1 \to P^1$ *is wrapping* $S^1$ *twice on itself.*

What about $P^2$? The psychological difficulty in visualizing the space is that one has to imagine two points on $S^2$ as one point. Note that $S^2$ may be divided into upper and lower halves, and any point in the lower half is identified with a point in the upper half. Therefore the map

$$\pi\colon \text{upper } S^2 \to P^2$$

is still onto. Within the upper half, the only remaining identification needed is the antipodal points on the equator $S^1$, or the boundary of the upper half. We already know the identification on the equator will simply produce $P^1$. Thus $P^2$ is obtained by gluing the boundary $S^1$ of $B^2$ (= upper $S^2$) to $P^1$ along the canonical projection $S^1 \to P^1$.

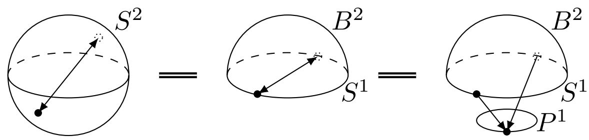



**Figure 6.3.4.** *a model of $P^2$*

The discussion about $P^2$ applies to higher dimensional cases. The map

$$\pi\colon \text{upper } S^n \to P^n$$

is onto, is one-to-one in the interior of the upper half sphere, and identifies antipodal points on the equator $S^{n-1}$ (the boundary of the upper half sphere). The identification on the equator gives precisely the projective space of one lower dimension $\pi\colon S^{n-1} \to P^{n-1}$. By identifying the upper half sphere with $B^n$, we get

$$P^n = P^{n-1} \cup_\pi B^n.$$

Thus $P^n$ is obtained by attaching an $n$-cell to $P^{n-1}$ along the canonical projection $\pi\colon S^{n-1} \to P^{n-1}$. This gives us a CW-structure of $P^n$

$$P^0 = \{pt\} \subset P^1 = S^1 \subset P^2 \subset \cdots \subset P^{n-1} \subset P^n,$$

which consists of one cell at each dimension.

Real projective spaces may be generalized in two ways. One is to consider complex or quaternionic numbers in place of real numbers. The other is the Grassmannians, which replaces the straight lines by linear subspaces of some other fixed dimension.

The *complex projective space* is

$$\begin{aligned}\mathbb{C}P^n &= \{\text{all (complex) straight lines in } \mathbb{C}^{n+1} \text{ passing through the origin}\}\\ &= \{\text{all complex 1-dimensional linear subspaces of } \mathbb{C}^{n+1}\}.\end{aligned}$$

To emphasize the underlying number systems, the real projective space is also denoted by $\mathbb{R}P^n$. Since complex 1-dimensional linear subspaces are the spans of nonzero complex vectors, we have

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C} - \{0\}} = \frac{\mathbb{C}^{n+1} - \{0\}}{v \sim \lambda v, \lambda \in \mathbb{C} - \{0\}},$$

Because two unit vectors span the same complex linear subspace if and only if one is a multiple of the other by a complex number of norm 1, we also have

$$\mathbb{C}P^n = \frac{S^{2n+1}}{S^1} = \frac{S^{2n+1}}{v \sim \lambda v, \lambda \in S^1},$$

where $S^{2n+1}$ is the unit sphere in $\mathbb{C}^{n+1}$, and $S^1 = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$ is the unit circle in the complex number plane. Thus while the real projective space is obtained by thinking of (antipodal) pairs of points as single points, the complex projective space is obtained by thinking of circles of points as single points.

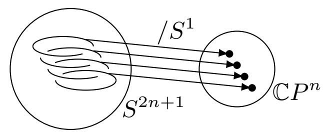



**Figure 6.3.5.** *$\mathbb{C}P^n$ is obtained from $S^{2n+1}$ by identifying circles to points.*

Since $S^1$ is only one circle, $\mathbb{C}P^0$ is a single point. To find $\mathbb{C}P^1$, start with

$$S^3 = \{(x, y) \subset \mathbb{C}^2 \colon |x|^2 + |y|^2 = 1\}.$$

The circle that $(x, y)$ belongs to is $[x, y] = \{(\lambda x, \lambda y) \colon |\lambda| = 1\}$, which represents one point in $\mathbb{C}P^1$. In other words, if we introduce the map

$$h[x, y] = \begin{cases} \dfrac{x}{y}, & \text{if } y \neq 0 \\ \infty, & \text{if } y = 0 \end{cases} : \mathbb{C}P^1 \to S^2 = \mathbb{C} \cup \{\infty\},$$

then $h[x, y] = h[z, w]$ if and only if $(x, y)$ and $(z, w)$ belong to the same circle. Therefore $h$ is a one-to-one correspondence. It turns out that $h$ is a homeomorphism. (This can be verified either directly or by using the fact that both $\mathbb{C}P^1$ and $S^2$ are compact Hausdorff spaces.) Thus we conclude that $\mathbb{C}P^1 = S^2$.

The general complex projective space has a CW-structure

$$\mathbb{C}P^0 = \{pt\} \subset \mathbb{C}P^1 = S^2 \subset \mathbb{C}P^2 \subset \cdots \subset \mathbb{C}P^{n-1} \subset \mathbb{C}P^n,$$

similar to the real one, with

$$\mathbb{C}P^k = \mathbb{C}P^{k-1} \cup_\pi B^{2k}$$

obtained by attaching a $2k$-cell to $\mathbb{C}P^{k-1}$ along the canonical projection $\pi \colon S^{2k-1} \to \mathbb{C}P^{k-1}$.

## 6.4 Euler Number

In Section 3.3, the Euler formula for the sphere and the plane were established. Some interesting applications of the formula were also discussed. From the CW-complex viewpoint, the vertices, edges, faces are nothing but 0-, 1-, and 2-dimensional simplices. Therefore the Euler formula may be extended by introducing the following concept.

**Definition 6.4.1.** The *Euler number* of a finite CW-complex $X$ is

$$\chi(X) = \sum (-1)^i (\text{number of } i\text{-cells in } X).$$

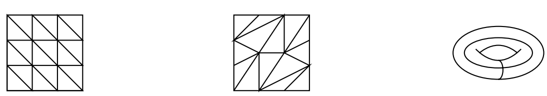
Since simplicial complexes are CW-complexes, the Euler number is also defined for simplicial complexes

$$\chi(K) = \sum (-1)^i (\text{number of } i\text{-simplices in } K).$$

A finite 0-dimensional CW-complex is finitely many points, and the Euler number is the number of points. Since the closed interval has a CW-structure given by two 0-cells (two ends) and one 1-cell (the interval itself), its Euler number is $2 - 1 = 1$. More generally, $\chi(\text{graph}) = (\text{number of vertices}) - (\text{number of edges})$. The following are more examples of CW-complexes and their Euler numbers.

| **CW-complex** | **0-cell** | **1-cell** | **2-cell** | **$k$-cell** | $\chi$ |
|---|---|---|---|---|---|
| circle $S^1$ | 1 | 1 | 0 | 0 | 0 |
| $S^n$ | 1 | 0 | 0 | $\begin{cases} 1, & \text{if } k = 0, n \\ 0, & \text{if } k \neq 0, n \end{cases}$ | $1 + (-1)^n$ |
| rectangle, disk | 1 | 1 | 1 | 0 for $k \geq 3$ | 1 |
| annulus | 2 | 3 | 1 | 0 for $k \geq 3$ | 0 |
| torus | 1 | 2 | 1 | 0 for $k \geq 3$ | 0 |
| $\mathbb{R}P^n$ | 1 | 1 | 1 | 1 for all $0 \leq k \leq n$ | $\begin{cases} 1, & \text{for even } n \\ 0, & \text{for odd } n \end{cases}$ |
| $\mathbb{C}P^n$ | 1 | 0 | 1 | $\begin{cases} 1, & \text{for even } k \\ 0, & \text{for odd } k \end{cases}$ | $n + 1$ |

A space (or polyhedron) can have many different CW-structures (or simplicial structures). Therefore, strictly speaking, the Euler number needs to be independent of the choice of the structures in order for the definition to be really rigorous. For example, the different triangulations and CW-structures of the torus in Figure 6.4.1 give the same Euler number 0. Moreover, Theorem 3.3.1 essentially says that a region on the plane without holes always has the Euler number 1, no matter how it is divided into smaller pieces by a connected graph. A rigorous proof of the independence of the Euler number on the choice of the CW-structure in general requires more advanced topology theory, such as the *homology theory*.



$9 - 27 + 18 = 0$ $\quad$ $7 - 21 + 14 = 0$ $\quad$ $1 - 2 + 1 = 0$

**Figure 6.4.1.** *The Euler number of the torus is always* 0.

The following are some useful properties of the Euler number.

**Lemma 6.4.2.** *If $X$ and $Y$ are finite CW-complexes, then*

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y),$$
$$\chi(X \times Y) = \chi(X)\chi(Y).$$

*Proof.* By $X \cup Y$, we mean a CW-complex, such that $X$, $Y$ and $X \cap Y$ are unions of cells in $X \cup Y$. (Thus $X$, $Y$ and $X \cap Y$ are *CW-subcomplexes* of $X \cup Y$.) Let $f_i(X)$ be the number of $i$-cells in $X$. Then we have

$$f_i(X \cup Y) = f_i(X) + f_i(Y) - f_i(X \cap Y).$$

The first equality can then be obtained by taking the alternating sum of both sides over all $i = 0, 1, 2, \dots$.

By $X \times Y$, we mean the product CW-complex, in which a cell is the product of a cell in $X$ and a cell in $Y$. Since $B^j \times B^k$ is homeomorphic to $B^{j+k}$, we get

$$f_i(X \times Y) = \sum_{j+k=i} f_j(X) f_k(Y).$$

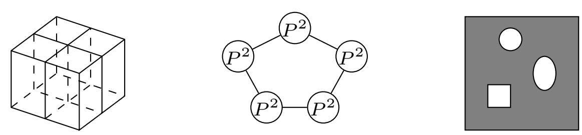
Then

$$\chi(X \times Y) = \sum_i (-1)^i \sum_{j+k=i} f_j(X) f_k(Y) = \sum_{j,k} (-1)^{j+k} f_j(X) f_k(Y)$$
$$= \left(\sum_j (-1)^j f_j(X)\right)\left(\sum_k (-1)^k f_k(Y)\right) = \chi(X)\chi(Y). \qquad \square$$

Exercise 6.4.1. Find the Euler numbers of the Möbius band, the Klein bottle and the $n$-ball $B^n$.

Exercise 6.4.2. Find the Euler number of the $n$-simplex $\Delta^n$.

Exercise 6.4.3. Find the Euler number of the spaces in Figure 6.4.2.




union of 4 boxes ring of 5 $P^n$ square with $n$ holes

**Figure 6.4.2.** *Find the Euler numbers.*

Exercise 6.4.4. Suppose a space $X = A \cup B \cup C$, with

$$A = B = C = \mathbb{C}P^2, \quad A \cap B = A \cap C = B \cap C = S^2, \quad A \cap B \cap C = \text{three points}.$$

Compute the Euler number of $X$.

Chapter 7

# Topological Properties

## 7.1 Hausdorff Space

**Definition 7.1.1.** A topological space $X$ is *Hausdorff* if for any $x \neq y$ in $X$, there are disjoint open subsets $U$ and $V$, such that $x \in U$ and $y \in V$.

In a Hausdorff space, any two points can be "separated" by disjoint open subsets. The open subsets in the definition may be replaced by subsets in topological basis, or subsets in local topological basis, or neighborhoods.

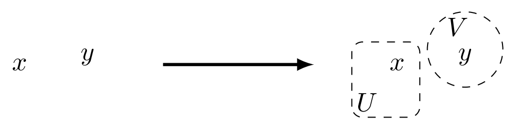



**Figure 7.1.1.** *Hausdorff property*

The Hausdorff property is clearly a topological property: If $X$ is a Hausdorff space, then any topological space homeomorphic to $X$ is also Hausdorff.

The Hausdorff property is one among a series of *separation axioms*.

- A topology is $T_0$ (or *Kolmogorov*[12]), if any two distinct points can be distinguished by an open subset: For any $x \neq y$, there is an open subset $U$, such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.

- A topology is $T_1$ (or *Fréchet*), if any point is separated from any distinct point by an open subset: For any $x \neq y$, there is an open subset $U$, such that $x \in U$ and $y \notin U$. It is easy to show that the condition is equivalent to any single point being a closed subset (see Exercise 4.5.22).

- A topology is $T_2$ (or *Hausdorff*), if any two distinct points are separated by disjoint open subsets.

- A topology is *regular*, if a point and a closed subset not containing the point can be separated by disjoint open subsets: For any closed $A$ and $x \notin A$, there are disjoint open subsets $U$ and $V$, such that $x \in U$ and $A \subset V$. A topology is $T_3$ if it is regular and Hausdorff.

- A topology is *normal*, if disjoint closed subsets can be separated by disjoint open subsets: For any disjoint closed subsets $A$ and $B$, there are disjoint open subsets $U$ and $V$, such that $A \subset U$ and $B \subset V$. A topology is $T_4$ if it is normal and Hausdorff.

**Example 7.1.1.** Among the four topologies on the two point space $\{1, 2\}$ in Example 4.3.4, only the discrete topology is Hausdorff. The other three are not Hausdorff.

[12] Andrei Nikolaevich Kolmogorov, born April 25, 1903 in Tambov, Russia, died October 20, 1987 in Moscow, USSR. Kolmogorov was a preeminent mathematician of the 20th century. He made fundamental contributions to may fields of mathematics, including probability, topology, mechanics and computational complexity.

**Example 7.1.2.** Among the topologies induced by the topological bases in Example 4.1.3, only $\mathbb{R}_{\hookleftarrow}$, $\mathbb{R}_{\hookrightarrow}$ and $\mathbb{R}_{\text{finite complement}}$ (induced by $\mathcal{B}_5$, $\mathcal{B}_6$ and $\mathcal{B}_9$) are non-Hausdorff.

**Example 7.1.3.** By taking $U = B\left(x, \dfrac{d(x,y)}{2}\right)$ and $V = B\left(y, \dfrac{d(x,y)}{2}\right)$, any metric space is Hausdorff (see Exercise 2.2.9).

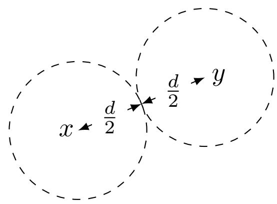



**Figure 7.1.2.** *Metric spaces are Hausdorff.*

**Example 7.1.4.** Suppose $f, g \in C[0,1]$ are different functions. Then $f(t_0) \neq g(t_0)$ for some $t_0 \in [0,1]$. For $\epsilon = \dfrac{|f(t_0) - g(t_0)|}{2}$, the subsets $U = B(f(t_0), t_0, \epsilon)$ and $V = B(g(t_0), t_0, \epsilon)$ are disjoint subsets containing $f$ and $g$ respectively. Therefore the pointwise convergence topology is Hausdorff.

**Example 7.1.5.** We prove that a topological space $X$ is Hausdorff if and only if the diagonal $\Delta(X)$ (see Example 5.4.3) is a closed subset of the product space $X \times X$.

By

$$\begin{aligned}
(x,y) \in U \times V &\iff x \in U,\ y \in V, \\
(x,y) \in X \times X - \Delta(X) &\iff x \neq y, \\
U \times V \subset X \times X - \Delta(X) &\iff U \cap V = \emptyset,
\end{aligned}$$

and the definition of product topology, we have

$$\begin{aligned}
&\Delta(X) \text{ is closed in } X \times X \\
\iff & X \times X - \Delta(X) \text{ is open in } X \times X \\
\iff & (x,y) \in X \times X - \Delta(X) \text{ implies } (x,y) \in U \times V \subset X \times X - \Delta(X), \\
& \text{for some open } U \subset X,\ V \subset Y \\
\iff & x \neq y \text{ implies } x \in U,\ y \in V,\ U \cap V = \emptyset, \text{ for some open } U \subset X,\ V \subset Y.
\end{aligned}$$

The last statement is the definition of Hausdorff property.

Exercise 7.1.1. Which topologies in Exercises 4.5.8, 4.5.9, 4.6.3 are Hausdorff?

Exercise 7.1.2. Prove that a subspace of $\mathbb{R}_{\hookleftarrow}$ is Hausdorff if and only if it is empty or contains only one number.

Exercise 7.1.3. Which subspaces of the line with two origins in Example 5.5.2 are Hausdorff?

Exercise 7.1.4. By taking the product of two of three topologies $\mathbb{R}_{(\text{—})}$, $\mathbb{R}_{[\text{—})}$, $\mathbb{R}_{(\text{—}}$, we get three topologies on $\mathbb{R}^2$. Which subspaces are Hausdorff?

1. $\{(x,y) \colon x+y \in \mathbb{Z}\}$.
2. $\{(x,y) \colon xy \in \mathbb{Z}\}$.
3. $\{(x,y) \colon x^2+y^2 \le 1\}$.

Exercise 7.1.5. Prove that the finite complement topology is Hausdorff if and only if the space is finite.

Exercise 7.1.6. Prove that in a Hausdorff space, any single point is a closed subset. However, there are non-Hausdorff spaces in which any single point is closed. This shows that $T_2$ implies $T_1$ but not vice versa.

Exercise 7.1.7. Prove that any finite Hausdorff space is discrete. In fact, any finite $T_1$ space is discrete.

Exercise 7.1.8. Suppose a topology is Hausdorff. Is a finer topology also Hausdorff? What about a coarser topology?

Exercise 7.1.9. Prove that any subspace of a Hausdorff space is Hausdorff. Then use the line with two origins in Example 5.5.2 to show that for Hausdorff subspaces $A, B \subset X$, the union $A \cup B$ and the closrue $\bar{A}$ are not necessarily Hausdorff.

Exercise 7.1.10. Prove that if $X$ and $Y$ are not empty, then $X \times Y$ is Hausdorff if and only if $X$ and $Y$ are Hausdorff.

Exercise 7.1.11. Prove that if $x_1, x_2, \dots, x_n$ are distinct points in a Hausdorff space, then there are mutually disjoint open subsets $U_1, U_2, \dots, U_n$ such that $x_i \in U_i$.

Exercise 7.1.12. Prove that in a Hausdorff space, any infinite open subset contains two disjoint open subsets, such that at least one is still infinite. Then further prove that any infinite Hausdorff space contains an infinite discrete subspace.

Exercise 7.1.13. Let $f, g \colon X \to Y$ be two continuous maps. Prove that if $Y$ is Hausdorff, then $A = \{x \colon f(x) = g(x)\}$ is a closed subset of $X$.

Exercise 7.1.14. Let $f \colon X \to Y$ be a continuous map. Prove that if $Y$ is Hausdorff, then the graph $\Gamma_f = \{(x, f(x)) \colon x \in X\}$ is a closed subset of $X \times Y$.

Exercise 7.1.15. A *cross section* of a continuous map $f \colon X \to Y$ is a continuous map $\sigma \colon Y \to X$ satisfying $f(\sigma(y)) = y$. In particular, the existence of the cross section implies that $f$ is onto.

1. Prove that the cross section map $\sigma$ is an embedding. In other words, the subspace $\sigma(Y)$ of $X$ is homeomorphic to $Y$.
2. Prove that if $X$ is Hausdorff, then $\sigma(Y)$ is a closed subset of $X$.
3. Suppose $A \subset X$ is a closed subset, such that for any $y \in Y$, the intersection $A \cap f^{-1}(y)$ consists of a single point which we denote by $\sigma(y)$. Prove that if $f$ maps

closed subsets to closed subsets, then the map $\sigma\colon Y \to X$ is continuous and is therefore a cross section of $f$.

Exercise 7.1.16. Suppose $X$ is a set, $Y_i$ are topological spaces, and $f_i\colon X \to Y_i$ are maps.

1. Prove that there is a coarsest topology $\mathcal{T}$ on $X$ such that all $f_i$ are continuous.
2. Prove that a necessary condition for $\mathcal{T}$ to be Hausdorff is that $x \neq x'$ in $X$ implies that $f_i(x) \neq f_i(x')$ for some $f_i$.
3. Prove that if $Y_i$ are Hausdorff, then the necessary condition in the second part is also sufficient.

Combined with Exercise 5.1.11, this gives a general explanation of Example 7.1.4.

Exercise 7.1.17. Prove that the following are equivalent conditions for a space to be regular.

1. If $x \in U$ and $U$ is open, then there is an open $V$, such that $x \in V$ and $\bar{V} \subset U$.
2. If $x \notin A$ and $A$ is closed, then there is an open $V$, such that $x \in V$ and $\bar{V} \cap A = \emptyset$.

Exercise 7.1.18. Prove that a Kolmogorov and regular space is Hausdorff.

Exercise 7.1.19. Suppose a topology is regular. Is a finer topology also regular? What about a coarser topology?

Exercise 7.1.20. Prove that the subspace of a regular space is regular.

Exercise 7.1.21. Prove that if $X$ and $Y$ are not empty, then $X \times Y$ is regular if and only if $X$ and $Y$ are regular.

## 7.2 Connected Space

Intuitively, a space is connected if it consists of only "one piece". To make the intuition more precise, we study how a space can be "separated" into two pieces.

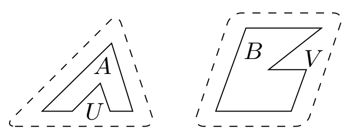



**Figure 7.2.1.** *A and B are separated by topology.*

In a Hausdorff space, distinct points are separated by disjoint open subsets. Similarly, two subsets $A$ and $B$ are *separated* if there are disjoint open subsets $U$ and $V$, such that $A \subset U$ and $B \subset V$. A *separation* of a topological space $X$ is then a decomposition $X = A \sqcup B$ into disjoint and separated subsets $A$ and $B$. In other words, we have

$$X = A \cup B, \quad A \subset U, \quad B \subset V, \quad U \cap V = \emptyset, \quad U, V \subset X \text{ are open.}$$

This implies $A = U$ and $B = V$, and leads to the following definition.

**Definition 7.2.1.** A *separation* of a topological space $X$ is a decomposition $X = A \sqcup B$ into disjoint, nonempty and open subsets $A$ and $B$. The space is *connected* if it has no separation.

The decomposition $X = A \sqcup B$ means $A$ and $B$ are each other's complement. This leads to the following equivalent descriptions of a separation.

1. $X = A \sqcup B$, where $A$ and $B$ are disjoint, nonempty and open.
2. $X = A \sqcup B$, where $A$ and $B$ are disjoint, nonempty and closed.
3. There is an open and closed subset $A \subset X$, such that $A \neq \emptyset, X$.

Consequently, we get equivalent criteria for a space to be connected.

1. If $X = A \sqcup B$, with $A$ and $B$ disjoint and open, then either $A$ or $B$ is empty.
2. If $X = A \sqcup B$, with $A$ and $B$ disjoint and closed, then either $A$ or $B$ is empty.
3. The only open and closed subsets of $X$ are $\emptyset$ and $X$.

The connected property is a topological property: If $X$ is connected, then any topological space homeomorphic to $X$ is also connected.

**Example 7.2.1.** Among the four topologies on the two point space $\{1, 2\}$ in Example 4.3.4, only the discrete topology is not connected. The other three are connected.

**Example 7.2.2.** The lower limit topology $\mathbb{R}_{[\ )}$ is not connected because $\mathbb{R} = (-\infty, 0) \cup [0, \infty)$ is a separation. The topology $\mathbb{R}_{(\_}$ is connected because the nontrivial open subsets are $(a, \infty)$ and the nontrivial closed subsets are $(-\infty, a]$, so that there is no nontrivial open and closed subset.

**Example 7.2.3.** Let $X$ be an infinite set with finite complement topology. Then nontrivial open subsets are infinite and nontrivial closed subsets are finite. Therefore there is no nontrivial open and closed subset, and the finite complement topology is connected.

Exercise 7.2.1. Is the trivial topology connected? Is the discrete topology connected?

Exercise 7.2.2. Which topologies in Exercises 4.5.8, 4.5.9, 4.6.3 are connected?

Exercise 7.2.3. Which 3 point spaces are connected? Is the topology on $\{1, 2, 3, 4\}$ in Example 4.3.5 connected?

Exercise 7.2.4. Which subsets of are connected in $\mathbb{R}_{[\ )}$, in $\mathbb{R}_{(\_}$, or in the Michael line in Exercise 4.4.6.

Exercise 7.2.5. Prove that $X$ is not connected if and only if there is a surjective continuous map from $X$ to the two point space with discrete topology.

Exercise 7.2.6. Suppose a topology is connected. Is a finer topology also connected? What about a coarser topology?

Exercise 7.2.7. Let $A$ and $B$ be disjoint subsets in a topological space $X$. Then $A$ and $B$ may be separated in $X$ in the sense that there are disjoint open subsets $U, V \subset X$, such that $A \subset U$ and $B \subset V$. Moreover, $A$ and $B$ may form a separation of $A \sqcup B$ in the sense that both $A$ and $B$ are open in the subspace $A \sqcup B$.

1. Prove that if $A$ and $B$ are separated in $X$, then $A$ and $B$ form a separation of $A \sqcup B$.
2. Suppose $X$ is a metric space. Use Exercise 5.3.17 to prove that if $A$ and $B$ form a separation of $A \sqcup B$, then $A$ and $B$ are separated in $X$.
3. Further use Exercise 5.3.17 to show that in general, it is possible for $A$ and $B$ to form a separation of $A \sqcup B$, but $A$ and $B$ are not separated in $X$.

**Theorem 7.2.2.** *Intervals in $\mathbb{R}$ are connected in the usual topology.*

*Proof.* Assume $A \subset [0,1]$ is open and closed with respect to the usual subspace topology and $0 \in A$. We will prove that $A = [0,1]$. This implies the theorem because if $A \subset [0,1]$ is open and closed and $0 \notin A$, then $B = [0,1] - A$ is open and closed and $0 \in B$. Therefore $B = [0,1]$, and $A = \emptyset$. Thus any open and closed subset $A \subset [0,1]$ must be either $[0,1]$ (in case $0 \in A$) or $\emptyset$ (in case $0 \notin A$).

Let
$$a = \sup\{x \colon [0,x] \subset A\}.$$
Since $0 \in A$ and $A$ is open, there is $\epsilon > 0$ such that $[0,\epsilon) \subset A$. This implies $a > 0$.

By the definition of $a$, there is an increasing sequence $x_n$, such that $[0,x_n] \subset A$ and $\lim x_n = a$. Since $A$ is closed, we have $a \in A$, so that $[0,a] = (\cup[0,x_n]) \cup \{a\} \subset A$.

If $a < 1$, then by $a \in A$ and $A$ open, we have $(a-\delta, a+\delta) \subset A$ for some $\delta$ satisfying $1 - a > \delta > 0$. Thus $[0,a+\delta) = [0,a] \cup (a-\delta,a+\delta) \subset A$, which by the definition of $a$ implies $a \geq a + \delta$. The contradiction implies that $a = 1$.

The connectedness of the other types of intervals can be proved in the similar way, or by using the connectedness of closed intervals and the second property of Theorem 7.2.3. ☐

**Theorem 7.2.3.** *Connectedness has the following properties.*

1. *If $A$ is connected and $A \subset B \subset \bar{A}$, then $B$ is connected.*
2. *If $A_i$ are connected and $\cap A_i \neq \emptyset$, then $\cup A_i$ is connected.*
3. *If $X$ is connected and $f \colon X \to Y$ is continuous, then $f(X)$ is connected.*
4. *Suppose $X$ and $Y$ are nonempty. Then $X \times Y$ is connected if and only if $X$ and $Y$ are connected.*

*Proof.* To prove the first property, consider a subset $C \subset B$ that is open and closed in the subspace topology of $B$. Since $C \cap A$ is open and closed in the subspace topology of $A$, and $A$ is connected, we have either $C \cap A = A$ or $C \cap A = \emptyset$.

Suppose $C \cap A = A$. Then $A \subset C$. This implies $\bar{A} \subset \bar{C}$, where the closure is taken in the whole space. Combined with $B \subset \bar{A}$, we have $B \subset \bar{C}$, which is equivalent to $B \cap \bar{C} = B$. Since $B \cap \bar{C}$ is the closure of $C$ in the subspace topology of $B$ (see Exercise 5.3.10), and $C$ is closed in the subspace topology of $B$, we get $C = B \cap \bar{C}$. Combined with $B \cap \bar{C} = B$, we further get $C = B$.

Suppose $C \cap A = \emptyset$, then the argument above may be repeated with $B - C$ in place of $C$. The end conclusion is $B - C = B$, which is equivalent to $C = \emptyset$. This completes the proof of the first property.

To prove the third property, consider a subset $A \subset f(X)$ that is open and closed in the subspace topology of $f(X)$. By the continuity of $f$, $f^{-1}(A)$ is open and closed in $X$. Since $X$ is connected, either $f^{-1}(A) = X$ or $f^{-1}(A) = \emptyset$. Since $A \subset f(X)$ implies $A = f(f^{-1}(A))$, we conclude that either $A = f(X)$ or $A = \emptyset$. This completes the proof of the third property.

The proof of the other properties are left as exercises. □

**Example 7.2.4.** As products of intervals, the (open or closed) square and cubes are connected, by the fourth property in Theorem 7.2.3. The balls are connected because they are homeomorphic to cubes. In fact, by the first property, any subset of $\mathbb{R}^n$ sandwiched between an open ball and its closure is connected. Moreover, the spaces in Figure 6.1.5 are all connected, by repeatedly applying the second property to the union of spaces homeomorphic to balls at various dimensions.

Sphere, torus, projective space, Klein bottle and their connected sums are connected because they are continuous images of the disk via the planar diagram.

**Example 7.2.5.** Suppose the 1-skeleton $X^1$ of a CW-complex $X$ is connected. Then the 2-skeleton $X^2$ is the union of $X^1$ and the images of disks $B^2$. The disks are connected, and their images are also connected by the third property in Theorem 7.2.3. Since the images of the disks share common points with the 1-skeleton (in fact, the images of the whole boundary circles are shared), the union $X^2$ of $X^1$ and the images of the disks is also connected.

Similar argument further shows that if the 1-skeleton of a CW-complex is connected, then the CW-complex is connected.

**Example 7.2.6.** The invertible real $n \times n$ matrices form a subset $GL(n, \mathbb{R})$ of $\mathbb{R}^{n^2}$. The determinant function $\det\colon GL(n, \mathbb{R}) \to \mathbb{R}$ is a continuous map, and has $\mathbb{R} - \{0\}$ as the image. Since $\mathbb{R} - \{0\}$ is not connected, it follows from the third property that $GL(n, \mathbb{R})$ is not connected.

Exercise 7.2.8. Prove the second property in Theorem 7.2.3.

Exercise 7.2.9. Prove the fourth property in Theorem 7.2.3.

1. Use the continuity of projections to prove the necessity.
2. For the sufficiency, first prove that for any $x \in X$ and $y \in Y$, $A_{x,y} = x \times Y \cup X \times y$ is connected. Then prove that $X \times Y = \cup_{y \in Y} A_{x,y}$ is connected.

Exercise 7.2.10. Prove that if a subset of $\mathbb{R}$ is not an interval, then it is not connected in the usual topology. So the only connected subsets in $\mathbb{R}_{(\,)}$ are intervals.

Exercise 7.2.11. Prove that a CW-complex is connected if and only if the 1-skeleton is connected.

Exercise 7.2.12. Show that the space $O(n, \mathbb{R}) = \{U : U^T U = I\} \subset \mathbb{R}^{n^2}$ of orthogonal matrices is not connected.

Exercise 7.2.13. Which subsets of the line with two origins in Example 5.5.2 are connected?

Exercise 7.2.14 (Intermediate Value Theorem). Suppose $X$ is connected and $f : X \to \mathbb{R}$ is a continuous map. Prove that for any $x_0, x_1 \in X$ and number $c$ satisfying $f(x_0) \leq c \leq f(x_1)$, there is $x \in X$ such that $f(x) = c$. Then use the result to prove that any invertible continuous map $f : (a, b) \to (c, d)$ is a homeomorphism.

Exercise 7.2.15 (Fixed Point Theorem). Suppose $f : [0, 1] \to [0, 1]$ is a continuous map. By considering $g(x) = f(x) - x$, prove that there is $x \in [0, 1]$, such that $f(x) = x$. What if $[0, 1]$ is replaced by $(0, 1)$?

Exercise 7.2.16. Use Exercise 7.2.4 to show that the only continuous maps from $\mathbb{R}_{(-)}$ to $\mathbb{R}_{[-)}$ or to the Michael line are the constant maps. The exercise may be compared with Exercise 5.1.2.

Exercise 7.2.17. Prove that for any continuous map $f : S^1 \to \mathbb{R}$, there is $x \in S^1$, such that $f(x) = f(-x)$.

Exercise 7.2.18. Suppose the only connected subsets in $X$ are single points. Prove that the only connected subsets of $X \times Y$ are $x \times B$, with $B \subset Y$ connected.

Exercise 7.2.19. Prove that if any two point subspace of a topological space $X$ is connected, then $X$ is also connected. What if two points are replaced by four points?

Exercise 7.2.20. Suppose $A_1, A_2, \ldots \subset X$ are connected subsets, such that $A_i \cap A_{i+1} \neq \emptyset$. Prove that $\cup A_i$ is connected.

Exercise 7.2.21. Suppose $A, A_i \subset X$ are connected subsets, such that $A \cap A_i \neq \emptyset$ for each $i$. Prove that $A \cup (\cup A_i)$ is connected.

Exercise 7.2.22. Let $Y \subset X$. Suppose both $X$ and $Y$ are connected and $X - Y = A \sqcup B$ is a separation. Use Exercise 5.3.12 to prove that $Y \cup A$ is connected.

## 7.3 Path Connected Space

**Definition 7.3.1.** A *path* connecting points $x$ and $y$ in a topological space $X$ is a continuous map $\gamma : [0, 1] \to X$, such that $\gamma(0) = x$ and $\gamma(1) = y$. The space $X$ is *path connected* if any two points can be connected by a path.

**Theorem 7.3.2.** *Path connected spaces are connected.*

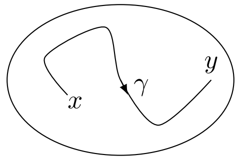



**Figure 7.3.1.** *path connected*

*Proof.* Fix a point $x \in X$. The path connected assumption implies that the space is the union of the images of the paths starting from $x$:

$$X = \cup_{\gamma(0)=x} \gamma[0, 1].$$

By Theorem 7.2.2 and the third property in Theorem 7.2.3, the image $\gamma[0, 1]$ is connected. Moreover, the images share $x$ as a common point. Then it follows from the second property in Theorem 7.2.3 that $X$ is connected. $\square$

**Example 7.3.1.** By constructing suitable paths in the disk or the square, we find the sphere, the torus, the projective space and the Klein bottle are all path connected.

**Example 7.3.2.** If we delete finitely many points from $\mathbb{R}^n$ $(n \geq 2)$, the sphere, the torus, the projective space or the Klein bottle, then we still get path connected spaces. If we delete a circle from $\mathbb{R}^n$ $(n \geq 3)$, then we also get a path connected space.

**Example 7.3.3.** A subset $A \subset \mathbb{R}^n$ is *convex* if

$$x, y \in A,\ 0 \leq t \leq 1 \implies \gamma(t) = (1-t)x + ty \in A.$$

Since $\gamma$ is the straight line segment connecting $x$ to $y$, any convex subset is path connected.

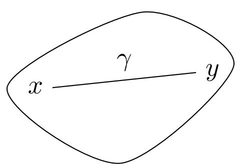



**Figure 7.3.2.** *convex subset*

**Example 7.3.4.** The space $X = \{1, 2\}$ with the topology $\mathcal{T} = \big\{\emptyset, \{1\}, \{1, 2\}\big\}$ is path connected because

$$\gamma(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 2, & \text{if } t = 1, \end{cases}$$

is a continuous path connecting 1 to 2.

**Example 7.3.5.** For any $f, g \in C[0,1]$, construct a path (we use $s$ to avoid confusion with the variable $t$ implicit in $f(t)$ and $g(t)$)

$$\gamma(s) = (1-s)f + sg \colon [0,1] \to C[0,1].$$

Note that for each fixed $s$, $\gamma(s)$ is a point in $C[0,1]$. The inequality

$$\begin{aligned} d_\infty(\gamma(s_1), \gamma(s_2)) &= \max_{t\in[0,1]} |[(1-s_1)f(t) + s_1 g(t)] - [(1-s_2)f(t) + s_2 g(t)]| \\ &= \max_{t\in[0,1]} |(s_1 - s_2)(f(t) - g(t))| \\ &= |s_1 - s_2| d_\infty(f, g) \end{aligned}$$

implies that, for any fixed $f$ and $g$, $\gamma$ is a continuous map into the $L_\infty$-topology. This shows that $C[0,1]_{L_\infty}$ is path connected.

**Example 7.3.6.** The subset

$$A = (0,1] \times 0 \cup \left( \cup_{n=1}^\infty \frac{1}{n} \times [0,1] \right) \subset \mathbb{R}^2$$

is path connected and is therefore also connected. The subset $B = A \cup \{(0,1)\}$ is still connected by $A \subset B \subset \bar{A}$ and the first property in Theorem 7.2.3. However, $B$ is not path connected because there is no path connecting the point $(0,1)$ to the other points of $B$.

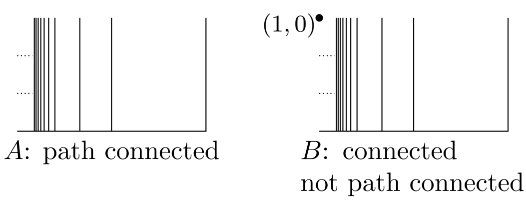



**Figure 7.3.3.** *Connected does not imply path connected.*

**Example 7.3.7.** Path connectedness can be used to show that $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic. Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. Then the restriction $f_| \colon \mathbb{R}^2 - \{f^{-1}(0)\} \to \mathbb{R} - \{0\}$ is also a homeomorphism. However, this contradicts the fact that $\mathbb{R}^2 - \{f^{-1}(0)\}$ is path connected and $\mathbb{R} - \{0\}$ is not.

**Example 7.3.8.** In Example 7.2.6, we see real invertible matrices do not form a connected space. The reason is that the determinant may take any value in $\mathbb{R} - \{0\}$, which is not connected. By taking only those invertible matrices with positive determinants, there is a hope that the space

$$GL_+(n, \mathbb{R}) = \{M \colon M \text{ is an } n \times n \text{ matrix and } \det M > 0\}$$

is connected. In fact, we will prove that $GL_+(n, \mathbb{R})$ is path connected.

Any invertible matrix $M$ can be written as $M = AU$, where $A$ is a positive definite matrix[13] and $U$ is an orthogonal matrix[14]. Moreover, $\det M > 0$ implies $\det U = 1$.

For any positive determinant matrix $M$, we will construct a *positive matrix path* $A(t)$ and an *orthogonal matrix path* $U(t)$, such that $A(0) = U(0) = I$, $A(1) = A$, $U(1) = U$. Then $M(t) = A(t)U(t)$ is a *matrix path* in $GL_+(n, \mathbb{R})$ connecting the identity matrix $M(0) = I$ to $M(1) = M$. This will imply that $GL_+(n, \mathbb{R})$ is path connected.

Since any symmetric matrix can be diagonalized, we have $A = PDP^{-1}$, where $P$ is an orthogonal matrix, $D$ is a diagonal matrix, with eigenvalues $\lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_n > 0$ of $A$ along the diagonal. Then

$$A(t) = P \begin{pmatrix} t\lambda_1 + (1-t) & 0 & \cdots & 0 \\ 0 & t\lambda_2 + (1-t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t\lambda_n + (1-t) \end{pmatrix} P^{-1}$$

is a positive matrix path connecting $I$ to $A$.

Since any orthogonal transformation is a combination of rotations, identities, and reflections with respect to an orthonormal basis, we have $U = QRQ^{-1}$, where $Q$ is an orthogonal matrix, and

$$R = \begin{pmatrix} R_{\theta_1} & 0 & \cdots & 0 & 0 \\ 0 & R_{\theta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_m & 0 \\ 0 & 0 & \cdots & 0 & I_n \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

By $\det R = \det U = 1$, $m$ is even. Therefore $-I_m$ may be considered as $\dfrac{m}{2}$ blocks of the matrix

$$R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$U(t) = Q \begin{pmatrix} R_{t\theta_1} & & & & & \\ & R_{t\theta_2} & & & & \\ & & \ddots & & & \\ & & & R_{t\pi} & & \\ & & & & \ddots & \\ & & & & & I_n \end{pmatrix} Q^{-1}$$

is an orthogonal matrix path connecting $I$ to $U$.

Exercise 7.3.1. Is the trivial topology path connected? Is the discrete topology path connected? What about the finite complement topology?

Exercise 7.3.2. Suppose a topology is path connected. Is a finer topology also path connected? What about a coarser topology?

[13] A positive definite matrix is a symmetric matrix with all eigenvalues positive.

[14] A square matrix $U$ is orthogonal if $U^T U = I$. The determinant of any orthogonal matrix is 1 or $-1$.

Exercise 7.3.3. Which properties in Theorem 7.2.3 still hold for path connected spaces?

Exercise 7.3.4. Prove that if there is $x_0 \in X$, such that for any $x \in X$, there is a path connecting $x$ to $x_0$, then $X$ is path connected. This fact is used in Example 7.3.8. Moreover, a subset $A \subset \mathbb{R}^n$ is *star-like* if there is a fixed point $x_0 \in A$, such that

$$x \in A,\ 0 \le t \le 1 \implies \gamma(t) = (1-t)x + tx_0 \in A.$$

Star-like subsets are path connected.

Exercise 7.3.5. Which subsets of the Euclidean space with the usual topology are connected or path connected?

1. $\{(x,y) \colon x^4 + y^4 = 1\}$.
2. $\{(x,y) \colon x^4 - y^4 = 1\}$.
3. $\{(x,y,z) \colon x^3 + y^3 = z^2\}$.
4. $\{(x,y,z) \colon x^3 + y^3 = z^2, x^4 + y^4 \le 1\}$.
5. $\{(x,y,z) \colon x^3 + y^3 = z^2, x + y > 0\}$.
6. $\{(x,y,z) \colon x^4 + y^4 + z^4 \le 4, xyz = 1\}$.
7. $\{(x,y,z,w) \colon x^2 + y^2 + z^2 - w^2 = 1\}$.
8. $\{(x,y,z,w) \colon x^2 + y^2 - z^2 - w^2 = 1\}$.
9. $\{(x,y,z,w) \colon x^2 - y^2 - z^2 - w^2 = 1\}$.
10. $\{(x_0, x_1, \dots, x_n) \colon x_0^2 \le 1 + x_1^2 + \cdots + x_n^2\}$.
11. $\{(x_0, x_1, \dots, x_n) \colon x_0^2 \ge 1 + x_1^2 + \cdots + x_n^2\}$.
12. $\mathbb{Q} \times \mathbb{R}$.
13. $\mathbb{Q} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Q}$.
14. $\mathbb{R}^n - \mathbb{Q}^n$.
15. Topologist's sine curve in Exercise 4.6.2.
16. Closure of topologist's sine curve.
17. Complement of topologist's sine curve.
18. Topologist's sine curve union with the origin.
19. Special linear group $SL(n,\mathbb{R}) = \{M \colon \det M = 1\}$.
20. Orthogonal group $O(n,\mathbb{R}) = \{M \colon M^T M = 1\}$.
21. Special orthogonal group $SO(n,\mathbb{R}) = \{M \colon M^T M = 1 \text{ and } \det M = 1\}$.

Exercise 7.3.6. Prove that if $A \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2 - A$ is path connected.

Exercise 7.3.7. Prove that with the usual topology, the spaces $\mathbb{R}^2$, $\mathbb{R}$ and $S^1$ are not homeomorphic to each other.

Exercise 7.3.8. Which topologies in Exercises 4.5.8, 4.5.9, 4.6.3 are path connected?

Exercise 7.3.9. Study the connectivity and path connectivity of the subsets in Exercise 7.1.4 with respect to the topology $\mathbb{R}^2_{[\,)}$.

Exercise 7.3.10. Prove that any subset of $\mathbb{R}_{\leftarrow}$ is path connected.

Exercise 7.3.11. Prove that the straight line in Example 7.3.5 is also continuous in $L_1$- and pointwise convergence topologies. In particular, the two topologies are also path connected. Then use Exercise 5.4.4 to show that $\{(f,g)\colon f(0) \geq g(0)\}$ is not an open subset of $C[0,1]_{\text{pt conv}} \times C[0,1]_{\text{pt conv}}$.

Exercise 7.3.12. Which subsets are connected or path connected, with respect to $L_\infty$-, $L_1$- and pointwise convergence topologies?

1. $\{f \in C[0,1]\colon f(0) > f(1)\}$.
2. $\{f \in C[0,1]\colon f(0) \neq f(1)\}$.
3. $\{f \in C[0,1]\colon |f(0)| > |f(1)|\}$.
4. $\{f \in C[0,1]\colon |f(0)| \geq |f(1)|\}$.

Exercise 7.3.13. Prove that the following are equivalent for a CW-complex.

1. The CW-complex is connected.
2. The CW-complex is path connected.
3. The 1-skeleton of the CW-complex is path connected.

Exercise 7.3.14. A topological space $X$ is *locally path connected* if for any open neighborhood $U$ of $x$, there is an open and path connected subset $V$ satisfying $x \in V \subset U$.

1. Prove that open subsets of $\mathbb{R}^n$ are locally path connected.
2. Fix a point $x$ in a locally path connected space $X$. Prove that the subset
$$\{y \in X\colon \text{There is a path in } X \text{ connecting } x \text{ and } y\}$$
is open and closed.
3. Prove that a locally path connected space is connected if and only if it is path connected.

The last statement is the partial converse of Theorem 7.3.2.

Exercise 7.3.15. The following steps prove that a finite topological space is connected if and only if it is path connected.

1. Suppose a point has a minimal open subset containing it. In other words, we have $x \in U$, with $U$ open, such that $x \in V$ for open $V$ implies $U \subset V$. Prove that the minimal open subset $U$ is path connected.
2. Prove that any finite topological space is locally path connected.

Exercise 7.3.16. Prove that under either of the following assumptions, a space is connected if and only if it is path connected.

1. For any open neighborhood $U$ of any $x$, there is a path connected neighborhood $N$ of $x$ satisfying $x \in N \subset U$. The subtle point here is that $N$ is not necessarily open.
2. Any $x$ has a neighborhood $N$, such that any $y \in N$ can be connected to $x$ by a path in $X$.

What is the relation between the assumptions and the locally connected property?

## 7.4 Connected Component

A subset $A$ of a topological space $X$ is a *connected component* if it is a maximal connected subset of $X$:

- $A$ is connected.
- If $A \subset B \subset X$ and $B$ is connected, then $A = B$.

The concept of *path connected component* can be similarly defined.

By the first property in Theorem 7.2.3, connected components are closed. Since the property no longer holds for path connected spaces, path connected components are not necessarily closed. Any two connected components are either identical or disjoint. To see this, suppose $A$ and $B$ are connected components and $A \cap B \neq \emptyset$. Then by the second property in Theorem 7.2.3, $A \cup B$ is connected. Since $A$ and $B$ are maximal connected, we must have $A = A \cup B = B$.

For any $x \in X$, the subset

$$C_x = \cup\{A \subset X \colon A \text{ is connected and } x \in A\}$$

is connected by the second property in Theorem 7.2.3. Moreover, the way $C_x$ is constructed implies that it is the biggest connected subset containing $x$. Therefore $C_x$ is the connected component containing $x$.

**Proposition 7.4.1.** *Any topological space is the disjoint union of connected components. Moreover, the connected components are closed subsets.*

By the similar reason, any topological space is also the disjoint union of path connected components.

The connected component is a topological concept. A homeomorphism will take connected components in one space to the connected components in the other one. In particular, the number of connected components is also a topological invariant.

**Example 7.4.1.** Consider the topology on $\{1,2,3,4\}$ in Example 4.3.5. The subsets $\{1,2,3\}$ and $\{4\}$ are connected. Since $\{1,2,3\} \cup \{4\}$ is a separation of the whole space, we conclude that $\{1,2,3\}$ and $\{4\}$ are the two connected components.

**Example 7.4.2.** We have $GL(n,\mathbb{R}) = GL_+(n,\mathbb{R}) \cup GL_-(n,\mathbb{R})$, where $GL$, $GL_+$ and $GL_-$ respectively denote invertible matrices, positive determinant matrices and negative determinant matrices. By Example 7.3.8, $GL_+(n,\mathbb{R})$ is connected. Let

$$D = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $\det D = -1$ and $M \mapsto DM$ is a homeomorphism between $GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$. Therefore $GL_-(n,\mathbb{R})$ is also connected. On the other hand, by Example 7.2.6, $GL(n,\mathbb{R})$

is not connected. Therefore we conclude that $GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ are the two connected components of $GL(n,\mathbb{R})$.

**Example 7.4.3.** We try to classify the alphabets $A$ through $G$ by homeomorphism. It is easy to see that $E$ and $F$ are homeomorphic, and $C$ and $G$ are also homeomorphic. Thus the classification problem is reduced to $A,B,C,D,E$. We consider the possible number $\nu$ of connected components after deleting one point:

$$\nu(A)=\{1,2\},\quad \nu(B)=\{1,2\},\quad \nu(C)=\{1,2\},\quad \nu(D)=\{1\},\quad \nu(E)=\{1,2,3\}.$$

This shows that $D,E$ are not homeomorphic, and they are not homeomorphic to $A,B,C$.

To further compare $A,B,C$, we consider the number $\lambda$ of points $x$, such that $X-x$ is connected:

$$\lambda(A)=\infty,\quad \lambda(B)=\infty,\quad \lambda(C)=2.$$

We also consider the number $\mu$ of points $x$, such that $X-x$ is not connected:

$$\mu(A)=\infty,\quad \mu(B)=1,\quad \mu(C)=\infty.$$

Since homeomorphic spaces must have the same $\lambda$ and $\mu$, we conclude that $A,B,C,D,E$ are mutually not homeomorphic.

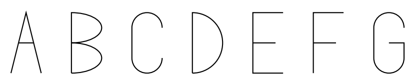



**Figure 7.4.1.** *alphabets*

Exercise 7.4.1. Find connected components and path connected components of $\mathbb{R}$ with respect to the topological bases in Example 4.1.3 and the Michael line in Exercise 4.4.6. Then prove that any continuous map from a connected space to $\mathbb{R}_{[\,)}$ or the Michael line must be a constant.

Exercise 7.4.2. Find connected components of the spaces in Exercises 4.5.8, 4.5.9, 4.6.3, 7.3.5, 7.3.12.

Exercise 7.4.3. Classify the objects in Figure 7.4.2 by homeomorphism.

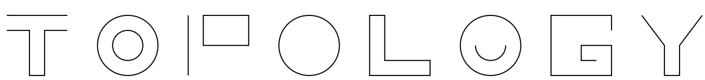


**Figure 7.4.2.** *topology*

Exercise 7.4.4. Let $\nu_n(X)$ be the possible number of connected components in $X-F$, where $F$ is a subset of $n$ points in $X$. If two connected graphs have the same $\nu_n$ for all $n$, is it true that the two graphs are homeomorphic?

Exercise 7.4.5. Prove that if $X$ has finitely many connected components, then any connected component is an open subset. Which subsets in such a topological space are open and closed?

Exercise 7.4.6. In a topological space, define $x \sim y$ if $x$ and $y$ can be connected by a path. Prove that $\sim$ is an equivalence relation, and the equivalence classes are the path connected components.

Exercise 7.4.7. Prove that any connected component is the union of some path connected components.

Exercise 7.4.8. A topological space $X$ is *locally connected* if for any neighborhood of $x$ contains a connected neighborhood of $x$.

1. Prove that any connected component of a locally connected space is open.
2. Prove that if any connected component of any open subset of a topological space is open, then the space is locally connected.
3. Let $f: X \to Y$ be onto and let $Y$ have the quotient topology. Prove that if $X$ is locally connected, so is $Y$.

Exercise 7.4.9. Prove that for locally path connected spaces (see Exercise 7.3.14), connected components and path connected components are the same. What if the locally path connected condition is replaced by the condition in Exercise 7.3.16?

## 7.5 Compact Space

Historically, the concept of compactness was quite elusive. For a long time, compactness of subsets of Euclidean spaces meant closed and bounded. It was also understood that compactness for metric spaces means that any sequence has a convergent subsequence. However, straightforward adoptions of these properties to general topological spaces were either not appropriate or yield non-equivalent definitions of compactness. The dilemma was even more serious in the days when the concept of topology was still ambiguous. After gaining lots of insights, mathematicians reached consensus and settled on the following definition.

**Definition 7.5.1.** A topological space is *compact* if any open cover has a finite subcover.

We need to unwrap the meaning of the definition bit by bit. Let $A \subset X$ be a subset. A *cover* of $A$ is a collection $\mathcal{U} = \{U_i\}$ of subsets of $X$, such that $A \subset \cup U_i$. For the special case $A = X$, the cover means $X = \cup U_i$. If $X$ is a topological space and all the subsets $U_i$ are open, then $\mathcal{U}$ is an *open cover*.

A subset $A$ of a topological space $X$ may be covered by open subsets in two ways. First we have the cover by open subsets of $X$ as given above. Second we have the cover by open subsets of $A$ in the subspace topology. Given a cover $\mathcal{U} = \{U_i\}$ by open subsets $U_i \subset X$, we get a cover $\mathcal{V} = \{U_i \cap A\}$ by open subsets $U_i \cap A \subset A$ in the subspace topology. Conversely, a cover $\mathcal{V}$ by open subsets of $A$ is given by

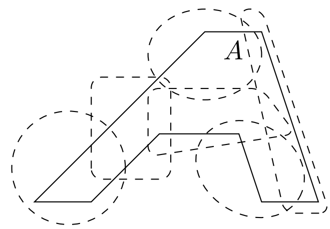



**Figure 7.5.1.** *open cover*

$V_i = U_i \cap A$ for some open subsets $U_i \subset X$. Then the collection $\mathcal{U} = \{U_i\}$ is a cover of $A$ by open subsets of $X$. The relation between the two types of open covers allows us to use either in the discussion.

**Example 7.5.1.** Let $X$ be all students in the university. Let $\mathcal{U}$ be the collection of all classes. Then $\mathcal{U}$ is a cover of $X$ if every student takes some classes.

**Example 7.5.2.** For any set $X$, $\mathcal{U} = \{\{x\} : x \in X\}$ is a cover. The cover is an open cover if and only if the topology is discrete.

**Example 7.5.3.** The following are covers of $\mathbb{R}$:

$$\mathcal{U}_1 = \{(-n, n) : n \in \mathbb{N}\},$$
$$\mathcal{U}_2 = \{[n, \infty) : n \in \mathbb{Z}\}.$$

The following are covers of $[0, 1]$ by subsets of $\mathbb{R}$:

$$\mathcal{U}_3 = \{(a - 0.1, a + 0.1) : 0 \le a \le 1\},$$
$$\mathcal{U}_4 = \{(a - 0.1, a + 0.1) : a = 0, 0.1, 0.2, \ldots, 1\}.$$

These are also covers of $(0, 1]$ by subsets of $\mathbb{R}$. Let $a_n$ be a strictly decreasing sequence with $\lim a_n = 0$ ($a_n = \dfrac{1}{n}$, for example), and let $\epsilon$ be any positive number. Then the following are covers of $(0, 1]$:

$$\mathcal{U}_5 = \{(a_n, 1] : n \in \mathbb{N}\},$$
$$\mathcal{U}_6 = \{(a_n, 1] : n \in \mathbb{N}\} \cup \{[0, \epsilon)\}.$$

Considering the topology, $\mathcal{U}_1$ is an open cover of $\mathbb{R}$ in the usual as well as the lower limit topologies. $\mathcal{U}_2$ is open in the lower limit topology and not open in the usual topology. $\mathcal{U}_3$ and $\mathcal{U}_4$ are covers by open subsets in $\mathbb{R}_{(\ )}$. $\mathcal{U}_5$ is a cover of $(0, 1]$ by the open subsets in $(0, 1]_{(\ )}$. $\mathcal{U}_6$ is a cover of $(0, 1]$ by the open subsets in $[0, 1]_{(\ )}$.

**Example 7.5.4.** For a fixed $\epsilon > 0$, $\mathcal{U} = \{B((m, n), \epsilon) : m, n \in \mathbb{Z}\}$ is the collection of open Euclidean balls with integer centers and radius $\epsilon$. It covers $\mathbb{R}^2$ if and only if $\epsilon > \dfrac{1}{\sqrt{2}}$. The cover is open in the usual topology.

**Example 7.5.5.** In a metric space $X$, for any fixed $\epsilon > 0$, the collection $\{B(x,\epsilon)\colon x \in X\}$ of all balls of radius $\epsilon$ is an open cover.

On the other hand, for any fixed $x \in X$, the collection $\{B(x,\epsilon)\colon \epsilon > 0\}$ of all balls centered at $x$ is also an open cover.

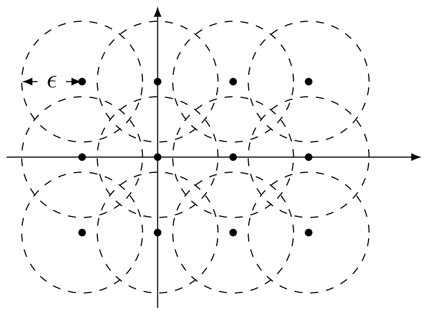



**Figure 7.5.2.** *open cover of* $\mathbb{R}^2$ *in case* $\epsilon > \dfrac{1}{\sqrt{2}}$

**Example 7.5.6.** Let $\mathcal{B}$ be a topological basis of $X$ and let $A \subset X$ be a subset. For each $a \in A$, choose $B_a \in \mathcal{B}$ satisfying $a \in B_a$. Then $\{B_a\colon a \in A\}$ is an open cover of $A$.

Exercise 7.5.1. How big should $\epsilon$ be in order for all the open Euclidean balls in $\mathbb{R}^n$ with integer centers and radius $\epsilon$ form an open cover? What about the $L_p$-metric?

Exercise 7.5.2. Which are open covers of $\mathbb{R}_{[\,)}$?

1. $\{[a, a+1)\colon a \in \mathbb{Q}\}$.
2. $\{[n, n+1)\colon n \in \mathbb{Z}\}$.
3. $\{(n, n+1)\colon n \in \mathbb{Z}\}$.
4. $\{(n, n+1]\colon n \in \mathbb{Z}\}$.
5. $\{(-\infty, n)\colon n \in \mathbb{N}\}$.
6. $\{(-\infty, n]\colon n \in \mathbb{N}\}$.

Exercise 7.5.3. Which are open covers of $C[0,1]$ in $L_1$-, $L_\infty$-, or pointwise convergence topology?

1. $\{B(0, 0, \epsilon)\colon \epsilon > 0\}$.
2. $\{B(a, 0, 1)\colon a \in \mathbb{R}\}$.
3. $\{B(1, t, 1)\colon t \in [0,1]\}$.
4. $\{B(a, t, 1)\colon a \in \mathbb{R}, t \in [0,1]\}$.
5. $\{\{f(t)\colon f(0) > a\}\colon a \in \mathbb{R}\}$.
6. $\{\{f(t)\colon |f(t)| < \epsilon \text{ for all } t\}\colon \epsilon > 0\}$.
7. $\left\{\left\{f(t)\colon \int_0^1 |f(t)|dt < n\right\}\colon n \in \mathbb{N}\right\}$.

If some subsets from a cover $\mathcal{U}$ still covers the space, then we get a *subcover*. Therefore the compactness means the following: If $\mathcal{U} = \{U_i\}$ is an open cover of $X$:

$$X = \cup U_i, \quad U_i \text{ open in } X,$$

then we can find finitely many $U_{i_1}, U_{i_2}, \ldots, U_{i_n} \in \mathcal{U}$ still covering $X$:

$$X = U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

The collection $\mathcal{V} = \{U_{i_1}, U_{i_2}, \dots, U_{i_n}\} \subset \mathcal{U}$ is a finite subcover. The compactness of a subset $A \subset X$ means

$$A \subset \cup U_i,\ U_i \text{ open in } X \implies A \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

**Example 7.5.7.** If a topological space contains only finitely many open subsets, then the space is compact. In particular, any finite topological space is compact. The trivial topology is also compact.

**Example 7.5.8.** For $(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k) \in \mathcal{U}_1$ in Example 7.5.3, we have

$$(-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k) = (-n, n) \neq \mathbb{R}, \quad n = \max\{n_1, n_2, \dots, n_k\}.$$

This shows that the open cover $\mathcal{U}_1$ has no finite subcover. Therefore the usual and the lower limit topologies on $\mathbb{R}$ are not compact.

The fact that the open cover $\mathcal{U}_3$ of $[0,1]$ and $(0,1]$ has a finite subcover $\mathcal{U}_4$ only shows that $\mathcal{U}_3$ is not a counterexample to the compactness of $[0,1]$ and $(0,1]$. The compactness requires the existence of finite subcover for *every* open cover.

By a reason similar to $\mathcal{U}_1$, the open cover $\mathcal{U}_5$ of $(0,1]$ has no finite subcover, which shows that $(0,1]$ is actually not compact in the usual topology, despite the other open cover $\mathcal{U}_3$ has a finite subcover.

For the open cover $\mathcal{U}_6$ of $[0,1]$ and $(0,1]$, we have $a_N < \epsilon$ for some $N$. Then $\{(a_N, 1], [0, \epsilon)\}$ is a subcover consisting of two subsets. This is again not a counterexample to the compactness of $[0,1]$ and $(0,1]$. In Theorem 7.5.3, we will prove that $[0,1]$ is compact in the usual topology.

**Example 7.5.9.** Consider the subset $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ of $\mathbb{R}$ with the usual topology. Let $\mathcal{U}$ be a cover of $A$ by open subsets of $\mathbb{R}$. Then $0 \in U$ for some $U \in \mathcal{U}$. Since $U$ is open, we have $\left(-\frac{1}{N}, \frac{1}{N}\right) \subset U$ for some big $N$. Moreover, for each $1 \leq n \leq N$, we have $\frac{1}{n} \in U_n$ for some $U_n \in \mathcal{U}$. Then $\mathcal{V} = \{U, U_1, U_2, \dots, U_N\}$ is a finite subcover of $A$. This proves that $A$ is compact.

**Example 7.5.10.** Suppose $X$ has finite complement topology. Suppose $\mathcal{U}$ is an open cover of $X$. If $X \neq \emptyset$, then some $U \in \mathcal{U}$ has the form $U = X - \{x_1, x_2, \dots, x_n\}$. Moreover, for each $1 \leq i \leq n$, we have $x_i \in U_i$ for some $U_i \in \mathcal{U}$. Then $\mathcal{V} = \{U, U_1, U_2, \dots, U_n\}$ covers $X$. This proves that the finite complement topology is compact.

Exercise 7.5.4. Explain why either type of open covers of a subset $A \subset X$ can be used for the definition of the compactness of $A$.

Exercise 7.5.5. Suppose a topology is compact. Is a finer topology also compact? What about a coarser topology?

Exercise 7.5.6. Determine the compactness of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, $\mathbb{Z}$ with respect to the topological bases in Example 4.1.3.

Exercise 7.5.7. A cover $\mathcal{V}$ is a *refinement* of a cover $\mathcal{U}$ if for any $V \in \mathcal{V}$, there is $U \in \mathcal{U}$, such that $V \subset U$.

1. Prove that if $\mathcal{V}$ has finite subcover, then $\mathcal{U}$ also has finite subcover.
2. Prove that if a topology is induced by a topological basis, then any open cover has a refinement consisting of subsets in the topological basis.
3. Prove that for the compactness, it is sufficient to verify that any cover by subsets in a topological basis has finite subcover.

**Theorem 7.5.2.** *Compactness has the following properties.*

1. *In a compact space, closed subsets are compact.*
2. *In a Hausdorff space, compact subsets are closed.*
3. *If $X$ is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.*
4. *If $X$ is compact and $Y$ is Hausdorff, then any bijective continuous map $f\colon X \to Y$ is a homeomorphism.*
5. *The union of two compact subsets is compact.*
6. *Suppose $X$ and $Y$ are nonempty. Then $X \times Y$ is compact if and only if $X$ and $Y$ are compact.*
7. *In a metric space, compact subsets are bounded and closed.*

*Proof.* Suppose $A$ is a closed subset in a compact space $X$. Suppose $\mathcal{U} = \{U_i\}$ is a collection of open subsets of $X$ covering $A$. Then $X = A \cup (X - A) \subset (\cup U_i) \cup (X - A)$. Since $A$ is closed, $X - A$ is open, and $\mathcal{U} \cup \{X - A\}$ is an open cover of $X$. By the compactness of $X$, we have

$$X \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \cup (X - A).$$

This implies

$$A \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n},$$

and we get a finite subcover of $A$ from $\mathcal{U}$. This proves the first property.

Suppose $A$ is a compact subset of a Hausdorff space $X$. For any $x \notin A$, we will find disjoint open subsets $U$ and $V$, such that $x \in U$ and $A \subset V$. Therefore $x \in U \subset X - A$. This implies $X - A$ is open, and the second property follows.

For each $a \in A$, we have $x \neq a$. Since $X$ is Hausdorff, there are disjoint open subsets $U_a$ and $V_a$, such that $x \in U_a$ and $a \in V_a$. By $A \subset \cup_{a \in A} V_a$ and $A$ compact, we have

$$A \subset V = V_{a_1} \cup V_{a_2} \cup \cdots \cup V_{a_n}$$

for finitely many $a_1, a_2, \ldots, a_n \in A$. We also have

$$x \in U = U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_n}.$$

By $U_a \cap V_a = \emptyset$, we further know that $U$ and $V$ are disjoint open subsets.

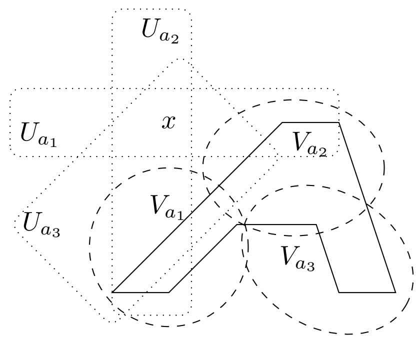



**Figure 7.5.3.** *The intersection of $U_{a_i}$ separates $x$ from $A$.*

For the third property, consider $f(X) \subset \cup U_i$ with $U_i$ open in $Y$. We have $X = \cup f^{-1}(U_i)$, where $f^{-1}(U_i)$ are open in $X$ by the continuity of $f$. Since $X$ is compact, we have

$$X = f^{-1}(U_{i_1}) \cup f^{-1}(U_{i_2}) \cup \cdots \cup f^{-1}(U_{i_n}) = f^{-1}(U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}).$$

This implies that

$$f(X) = f\left(f^{-1}(U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n})\right) \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

The key for the fourth property is the continuity of the inverse map $f^{-1}$, which is equivalent to $f$ sending closed subsets of $X$ to closed subsets of $Y$. Under the given assumption, we have

$$\begin{aligned} A \subset X \text{ is closed} &\implies A \text{ is compact} && (X \text{ is compact, and first property}) \\ &\implies f(A) \text{ is compact} && (f \text{ is continuous, and third property}) \\ &\implies f(A) \text{ is closed.} && (Y \text{ is Hausdorff, and second property}) \end{aligned}$$

Suppose $A, B \subset X$ are compact subsets. Any open cover $\mathcal{U}$ of $A \cup B$ by open subsets of $X$ is also an open cover of $A$. The compactness of $A$ implies that $A$ is covered by finitely many from $\mathcal{U}$. Similarly, $B$ is also covered by finitely many from $\mathcal{U}$. Combining the two finite subcovers together, we get a finite subcover of $A \cup B$. This completes the proof of the fifth property.

The sufficiency part of the sixth property follows from the third property and the continuity of the projections. For the necessity, assume $X$ and $Y$ are compact and consider an open cover $\mathcal{W}$ of $X \times Y$.

We first prove that for any fixed $x \in X$, there is an open subset $U^x$ containing $x$, such that $U^x \times Y$ is covered by finitely many open subsets in $\mathcal{W}$. For any $y \in Y$, we have $(x, y) \in W_y$ for some $W_y \in \mathcal{W}$. Since $W_y$ is open, we have $(x, y) \in U_y \times V_y \subset W_y$ for some open subsets $U_y \subset X$ and $V_y \subset Y$. Then the collection $\{V_y : y \in Y\}$ is an open cover of $Y$. Since $Y$ is compact, we get

$$Y = V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}.$$

The subset $U^x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$ contains $x$, is open in $X$, and satisfies

$$\begin{aligned} U^x \times Y &= (U^x \times V_{y_1}) \cup (U^x \times V_{y_2}) \cup \cdots \cup (U^x \times V_{y_n}) \\ &\subset (U_{y_1} \times V_{y_1}) \cup (U_{y_2} \times V_{y_2}) \cup \cdots \cup (U_{y_n} \times V_{y_n}) \\ &\subset W_{y_1} \cup W_{y_2} \cup \cdots \cup W_{y_n}. \end{aligned}$$

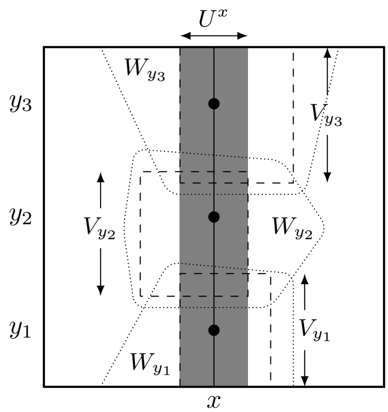



**Figure 7.5.4.** $x \in U^x$*, and* $U^x \times Y$ *is finitely covered.*

Since $\{U^x : x \in X\}$ is an open cover of the compact space $X$, we have

$$X = U^{x_1} \cup U^{x_2} \cup \cdots \cup U^{x_k}.$$

This implies

$$X \times Y = (U^{x_1} \times Y) \cup (U^{x_2} \times Y) \cup \cdots \cup (U^{x_k} \times Y).$$

Since each $U^{x_j} \times Y$ is covered by finitely many open subsets in $\mathcal{W}$, we conclude that $X \times Y$ is covered by finitely many open subsets in $\mathcal{W}$.

Finally, suppose $X$ is a metric space and $A \subset X$ is a compact subset. Fix one $x \in X$. Then the collection $\mathcal{U} = \{B(x, \epsilon) : \epsilon > 0\}$ of all balls centered at $x$ is an open cover of $A$. Since $A$ is compact, we have

$$A \subset B(x, \epsilon_1) \cup B(x, \epsilon_2) \cup \ldots \cup B(x, \epsilon_n) = B(x, \epsilon), \quad \epsilon = \max\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\},$$

which shows that $A$ is bounded. Moreover, since metric spaces are Hausdorff by Example 7.1.3, the compact subset $A$ is also closed, by the second property. This completes the proof of the seventh property. $\square$

The following characterizes compact subsets in the Euclidean spaces.

**Theorem 7.5.3** (Heine[15]-Borel[16] Theorem). *The closed interval* $[0,1]$ *is compact in the usual topology. In general, a subset of* $\mathbb{R}^n_{\text{usual}}$ *is compact if and only if it is bounded and closed.*

*Proof.* Suppose $\mathcal{U}$ is an open cover of $I_0 = [0,1]$, such that $[0,1]$ cannot be covered by finitely many subsets in $\mathcal{U}$. Then at least one of the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ of length $\frac{1}{2}$ cannot be covered by finitely many subsets in $\mathcal{U}$. Denote this interval by $I_1$ and divide $I_1$ into a union of two intervals of length $\frac{1}{4}$. Since $I_1$ cannot be covered by finitely many subsets in $\mathcal{U}$, at least one of the two intervals of length $\frac{1}{4}$ cannot be covered by finitely many subsets in $\mathcal{U}$. Denote this interval by $I_2$ and further divide $I_2$ into two intervals of length $\frac{1}{8}$. Keep going, we find a sequence of closed intervals $I_0$, $I_1$, $I_2$, $\dots$, satisfying

1. $I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$.
2. The length of $I_n$ is $\frac{1}{2^n}$.
3. None of $I_n$ is covered by finitely many subsets in $\mathcal{U}$.

Choose one point $x_n$ in each interval $I_n$. The first two properties imply that $|x_m - x_n| < \frac{1}{2^N}$ for $m, n > N$. Therefore $\{x_n\}$ is a Cauchy sequence and must have a limit $x$. By the first property, we have $x_m \in I_n$ for all $m \geq n$. Since the limit depends only on $x_n$ for sufficiently big $n$, the closedness of $I_n$ implies that $x \in I_n$ for all $n$.

On the other hand, since $\mathcal{U}$ is an open cover of $[0,1]$, $x \in U$ for some $U \in \mathcal{U}$. Since $U$ is open, we have $\left(x - \frac{1}{2^{n-1}}, x + \frac{1}{2^{n-1}}\right) \subset U$ for some $n$. Then by $x \in I_n$ and the second property above, we have $I_n \subset U$. In particular, $I_n$ is covered by one open subset in $\mathcal{U}$. This contradicts the third property.

Thus the original assumption that $\mathcal{U}$ has no finite subcover is wrong. This completes the proof that $[0,1]$ is compact.

In general, a compact subset of the usual Euclidean space is bounded and closed by the seventh property in Theorem 7.5.2. Conversely, suppose $A$ is a bounded and closed subset of the usual Euclidean space. Then $A \subset [-R, R]^n$ for some big $R$. The interval $[-R, R]$ is compact because it is homeomorphic to $[0,1]$. By the sixth property in Theorem 7.5.2, $[-R, R]^n$ is also compact. Then by the first property in Theorem 7.5.2, the closed subset $A$ of $[-R, R]^n$ is also compact. $\square$

[15] Heinrich Eduard Heine, born March 15, 1821 in Berlin, Germany, died October 21, 1881 in Halle, Germany. In addition to the Heine-Borel theorem, Heine introduced the idea of uniform continuity.

[16] Félix Edouard Justin Émile Borel, born January 7, 1871 in Saint-Affrique, France, died February 3, 1956 in Paris France. Borel's measure theory was the beginning of the modern theory of functions of a real variable. He was French Minister of the Navy from 1925 to 1940.

**Example 7.5.11.** The spheres are compact because they are bounded and closed subsets of the Euclidean spaces. As continuous images of the spheres, the real and complex projective spaces are also compact.

As products of bounded closed intervals, the closed square and the closed cubes are compact. The closed balls are also compact because they are homeomorphic to the closed cubes. The torus and the Klein bottle are compact because they are continuous images of the closed square.

**Example 7.5.12.** Finite simplicial complexes are compact because they are unions of finitely many simplices, each being homeomorphic to a closed ball and therefore compact. Finite CW-complexes are compact because they are unions of the continuous images of finitely many closed balls.

**Example 7.5.13.** The following subsets of $\mathbb{R}^3$

$$A_1 = \{(x,y,z)\colon x^4+y^4+2z^4 = 1\}, \qquad A_2 = \{(x,y,z)\colon x^4+y^4+2z^4 \le 1\},$$
$$A_3 = \{(x,y,z)\colon x^4+y^4+2z^4 < 1\}, \qquad A_4 = \{(x,y,z)\colon x^4+y^4+2z^4 \ge 1\},$$

can be expressed as $A_1 = f^{-1}(1)$, $A_2 = f^{-1}[0,1]$, $A_3 = f^{-1}(-\infty,1)$, $A_4 = f^{-1}[1,\infty)$, where $f(x,y,z) = x^4+y^4+2z^4$. By the continuity of $f$, $A_1$ and $A_2$ are closed. Moreover, the points in $A_1$ and $A_2$ satisfy $|x| \le 1$, $|y| \le 1$, $|z| \le 1$. Therefore $A_1$ and $A_2$ are also bounded. By Theorem 7.5.3, $A_1$ and $A_2$ are compact.

The subset $A_3$ is open. If $A_3$ were closed, then by the connectedness of $\mathbb{R}^3$, $A_3$ must be either empty or the whole $\mathbb{R}^3$. Since this is not the case, $A_3$ is not closed. Consequently, $A_3$ is not compact. Finally, $A_4$ is not compact because it is not bounded.

**Example 7.5.14.** As a subset of $\mathbb{R}^{n^2}$, the invertible matrices $GL(n,\mathbb{R})$ is not compact because it is not bounded. On the other hand, the orthogonal matrices $O(n) = f^{-1}(I)$, where $f(M) = M^TM$ is a continuous map from $n\times n$ matrices to $n\times n$ matrices. Therefore $O(n)$ is a closed subset of $\mathbb{R}^{n^2}$. Since the columns of an orthogonal matrix are vectors of length 1, $O(n)$ is bounded. Thus we conclude that $O(n)$ is compact.

Exercise 7.5.8. Which topologies in Exercises 4.5.8, 4.5.9, 4.6.3, 7.3.5 are compact?

Exercise 7.5.9. Prove that a subset $A \subset \mathbb{R}$ is compact in $\mathbb{R}_{\leftarrow}$ if and only if $\inf A$ is a finite number and $\inf A \in A$.

Exercise 7.5.10. The compactness of $[0,1]$ can be proved by another method. Let $\mathcal{U}$ be an open cover of $[0,1]$. Let

$$A = \{x\colon [0,x] \text{ is covered by finitely many open subsets in } \mathcal{U}\}.$$

Prove that $\sup A \in A$ and $\sup A = 1$.

Exercise 7.5.11. What are the compact subsets of $\mathbb{R}$ in the lower limit topology?

1. Suppose a subset $A \subset \mathbb{R}$ is compact in the lower limit topology. Prove that $A$ is compact in the usual topology. Moreover, prove that if $a_n \in A$ is strictly increasing, then $\lim a_n \notin A$.

2. Prove that the property $\lim a_n \notin A$ for any strictly increasing $a_n \in A$ is equivalent to the following: For any $a \in A$, there is $\epsilon > 0$, such that $(a - \epsilon, a) \cap A = \emptyset$.
3. Suppose $A$ is compact in the usual topology and $\lim a_n \notin A$ for any strictly increasing sequence $a_n \in A$. For any cover $\{[a_i, b_i)\}$ of $A$ by topological basis in the lower limit topology, by modifying each $[a_i, b_i)$ in a suitable way, construct a suitable cover of $A$ by usual open intervals. Then prove that $\{[a_i, b_i)\}$ has a finite subcover.

Exercise 7.5.12 (Extreme Value Theorem). Prove that if $f$ is a continuous function on a compact space $X$, then there are $x_0, x_1 \in X$, such that $f(x_0) \leq f(x) \leq f(x_1)$ for any $x \in X$. In other words, $f$ reaches its minimum at $x_0$ and maximum at $x_1$. Moreover, use Exercise 7.5.9 to prove that any lower semi-continuous function (see Exercise 5.1.28) on a compact space must reach its minimum.

Exercise 7.5.13. In the metric space $X$, the distance $d(x, A)$ from a point $x$ to a subset $A$ is defined in Exercise 2.4.15. Prove that if $A$ is compact, then $d(x, A) = d(x, a)$ for some $a \in A$. What about the distance $d(A, B) = \sup_{x \in A, y \in B} d(x, y)$ between two compact subsets?

Exercise 7.5.14. The *diameter* of a bounded metric space $X$ is $\sup_{x,y \in X} d(x, y)$. Prove that if $X$ is compact, then the diameter is equal to the distance between two points in $X$.

Exercise 7.5.15. Prove that any compact metric space has a countable topological basis (i.e., second countable).

Exercise 7.5.16. Let $\mathcal{T}$ be a topology on $X$. Then $Y = X \cup \{a, b\}$ has a topology obtained by adding $Y$, $Y - a$ and $Y - b$ to $\mathcal{T}$. Show that the subsets $Y - a$ and $Y - b$ are compact, although the intersection $X = (Y - a) \cap (Y - b)$ may not be compact. Therefore the intersection of two compact subsets may not be compact.

Exercise 7.5.17. Let $X$ be a topological space. Let $\{1, 2\}$ have the topology $\big\{\emptyset, \{1\}, \{1, 2\}\big\}$. Then $Y = (X \times \{1, 2\}) \cup \{a\}$ has a topology obtained by adding $Y$ and $(X \times \{1\}) \cup \{a\}$ to the product topology on $X \times \{1, 2\}$.

1. Show that $(X \times \{1\}) \cup \{a\}$ is a compact subset.
2. Show that the closure of $(X \times \{1\}) \cup \{a\}$ is $Y$.
3. Prove that if $X$ is not compact, then $Y$ is not compact.

Therefore the closure of a compact subset may not be compact.

Exercise 7.5.18. Exercises 7.5.16 and 7.5.17 show that the intersection and the closure of compact subsets may not be compact. Could this happen in Hausdorff spaces or Fréchet spaces?

Exercise 7.5.19. Suppose $X$ is a regular space.

1. Prove that disjoint compact and closed subsets can be separated by disjoint open subsets: If $A$ is compact, $B$ is closed, and $A \cap B = \emptyset$, then there are open subsets $U$ and $V$, such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.
2. Prove that if $A \subset U$, $A$ is compact and $U$ is open, then there is an open subset $V$, such that $A \subset V$ and $\bar{V} \subset U$.

Exercise 7.5.20. Suppose $X$ is a metric space. Prove that if $K \subset U$, $K$ is compact, $U$ is open, then there is $\epsilon > 0$, such that the $\epsilon$-neighborhood of $K$ (see Exercise 2.3.15)

$$K^\epsilon = \{x \in X : d(x, k) < \epsilon \text{ for some } k \in K\}$$

is contained in $U$.

Exercise 7.5.21. Prove that if $A$ and $B$ are disjoint compact subsets in a Hausdorff space, then there are disjoint open subsets $U$ and $V$, such that $A \subset U$ and $B \subset V$.

Note that the proof of the second property in Theorem 7.5.2 already proves the special case $A$ is a single point, which implies that a compact Hausdorff space is regular. The exercise further implies that a compact Hausdorff space is normal.

Exercise 7.5.22. Prove that if $Y$ is compact, then the projection $X \times Y \to X$ maps closed subsets to closed subsets. This complements Exercise 5.4.11.

Exercise 7.5.23 (Closed Graph Theorem). Suppose $Y$ is compact Hausdorff. Prove that a map $f : X \to Y$ is continuous if and only if the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. This is the partial converse to Exercise 7.1.14. Moreover, show that the compact condition cannot be dropped.

Exercise 7.5.24. Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be an onto map taking closed subsets to closed subsets. Prove that if $f^{-1}(y)$ is compact for any $y \in Y$ and $Y$ is compact, then $X$ is compact.

Exercise 7.5.25. Let $X$ be an infinite set and let $Y$ be any set. Prove that

$$\mathcal{T} = \{X \times Y - F : F \cap (X \times y) \text{ is finite for any } y \in Y\} \cup \{\emptyset\}$$

is a topology on $X \times Y$.

1. Prove that the quotient topology on $Y$ induced by the projection $\pi_Y : (X \times Y)_{\mathcal{T}} \to Y$ is the trivial topology.
2. Prove that $\pi_Y^{-1}(y) = X \times y$ is compact for any $y \in Y$.

This example shows that closed subsets in Exercise 7.5.24 cannot be changed to open subsets. In case $Y$ is finite, the example also shows that we could have an onto continuous map $f : X \to Y$, with $X$, $Y$, $f^{-1}(y)$ compact, yet $f$ does not take closed subsets to closed subsets.

Exercise 7.5.26. A collection $\mathcal{A}$ of subsets has the *finite intersection property* if $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$ for any finitely many $A_1, A_2, \ldots, A_n \in \mathcal{A}$.

1. Prove that a topological space is compact if and only if for any collection $\mathcal{C}$ of closed subsets with finite intersection property, we have $\cap_{C \in \mathcal{C}} C \neq \emptyset$.
2. Prove that if a collection $\mathcal{K}$ of compact subsets in a Hausdorff space has finite intersection property, then $\cap_{K \in \mathcal{K}} K \neq \emptyset$.
3. Prove that if a collection $\mathcal{K}$ of compact subsets in a Hausdorff space has finite intersection property, and $U$ is an open subset containing $\cap_{K \in \mathcal{K}} K$, then $K_1 \cap K_2 \cap \cdots \cap K_n \subset U$ for finitely many $K_1, K_2, \ldots, K_n \in \mathcal{K}$.

Exercise 7.5.27. Suppose $X$ is a Hausdorff space, with topology $\mathcal{T}$. Let $X^+ = X \sqcup \{+\}$ be obtained by adding one point to $X$. Also denote $A^+ = A \sqcup \{+\}$ for subsets $A \subset X$.

1. Prove that $\mathcal{T}^+ = \mathcal{T} \cup \{(X - K)^+ \colon K \text{ is a compact subset of } X\}$ is a topology on $X^+$.

2. Prove that $X^+$ is compact.

The topological space $X^+$, with topology $\mathcal{T}^+$, is called the *one-point compactification* of $X$.

Exercise 7.5.28. Prove that the one-point compactification of the usual Euclidean space $\mathbb{R}^n$ is the usual sphere $S^n$.

Exercise 7.5.29. Prove that the following are equivalent for a Hausdorff space $X$.

1. The one-point compactification $X^+$ is Hausdorff.

2. Any open neighborhood of $x$ contains a compact neighborhood of $x$.

3. Any point has an open neighborhood with compact closure.

4. Any point has a compact neighborhood.

The second, third and fourth properties are three of several versions of *local compactness*. Show that the three properties may not be equivalent when $X$ is not Hausdorff. Moreover, study the properties of locally compact spaces (subspace still locally compact? for example).

Exercise 7.5.30. Suppose $Y$ is a compact Hausdorff space. Suppose $+ \in Y$ is a point and $X = Y - \{+\}$. Prove that if $Y = \bar{X}$, then $Y$ is the one-point compactification of $X$.

Exercise 7.5.31. Prove that the three versions of local compactness in Exercise 7.5.29 are the same for regular spaces.

Exercise 7.5.32. Suppose $X$ is a locally compact and regular space. Prove that if $A \subset U$, $A$ is compact, $U$ is open, then there is an open subset $V$, such that $A \subset V$, $\bar{V} \subset U$ and $\bar{V}$ is compact.

Compared with the second statement of Exercise 7.5.19, the local compactness allows us to further get $\bar{V}$ to be comapct.

Exercise 7.5.33. Prove that locally compact and Hausdorff spaces are regular.

## 7.6 Limit Point Compact Space

The following property is more intuitive and easier to understand than compactness as defined by open covers.

**Definition 7.6.1.** A topological space is *limit point compact* (or *Fréchet compact*, or has *Bolzano*[17]*-Weierstrass*[18] *property*) if any infinite subset has a limit point.

Both compactness by open covers and by limit points were first introduced by Aleksandrov[19] and Urysohn[20] in 1923, who called the properties “bicompact” and “compact”, respectively. The word “compact” was later shifted to mean compactness by open cover, leaving the compactness by limit points without a commonly agreed name.

**Theorem 7.6.2.** *Compact spaces are limit point compact. Conversely, limit point compact metric spaces are compact.*

*Proof.* Let $X$ be a compact space and let $A \subset X$ be a subset without limit points. We will prove $A$ is finite. This shows that compact implies limit point compact.

For any $x \in X$, since $x$ is not a limit point of $A$, there is an open subset $U_x$, such that $x \in U_x$ and $(A - x) \cap U_x = \emptyset$. This implies $A \cap U_x \subset \{x\}$. Since the collection $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of $X$, the compactness of $X$ implies that $X = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$. Therefore

$$A = A \cap X = (A \cap U_{x_1}) \cup (A \cap U_{x_2}) \cup \cdots \cup (A \cap U_{x_n}) \subset \{x_1, x_2, \dots, x_n\}.$$

Now let $X$ be a limit point compact metric space. To prove $X$ is compact, we first prove the following claim: For any open cover $\mathcal{U}$ of $X$, there is $\epsilon > 0$, such any ball $B(x, \epsilon)$ of radius $\epsilon$ is contained in some $U \in \mathcal{U}$. Suppose the property does not hold. We choose a positive sequence $\epsilon_n$ converging to 0 (take $\epsilon_n = \dfrac{1}{n}$, for example). Then for any $n$, there is a ball $B(x_n, \epsilon_n)$ not contained in any open subset in $\mathcal{U}$. If the subset $A = \{x_n : n \in \mathbb{N}\}$ is finite, then infinitely many $x_n$ are equal to some $x$. If the subset is infinite, then by the assumption, $A$ has a limit point $x$. In either case, for any $\epsilon > 0$, there is (sufficiently big) $n$, such that $\epsilon_n < \epsilon$ and $d(x_n, x) < \epsilon$.

Since $X$ is covered by $\mathcal{U}$, we have $x \in U$ for some $U \in \mathcal{U}$. Since $U$ is open, we have $B(x, \epsilon) \subset U$ for some $\epsilon > 0$. For this $\epsilon$, as argued above, we have $\epsilon_n < \dfrac{\epsilon}{2}$ and $d(x_n, x) < \dfrac{\epsilon}{2}$ for some $n$. This implies $B(x_n, \epsilon_n) \subset B(x, \epsilon) \subset U$, which contradicts

[17]Bernard Placidus Johann Nepomuk Bolzano, born October 5, 1781, died December 18, 1848 in Prague, Bohemia (now Czech). Bolzano is famous for his 1837 book “Theory of Science”. He insisted that many results which were thought “obvious” required rigorous proof and made fundamental contributions to the foundation of mathematics. He understood the need to redefine and enrich the concept of number itself and defined the Cauchy sequence four years before Cauchy’s work appeared.

[18]Karl Theodor Wilhelm Weierstrass, born October 31, 1815 in Ostenfelde, Westphalia (now Germany), died February 19, 1848 in Berlin, Germany. In 1864, he found a continuous but nowhere differentiable function. His lectures on analytic functions, elliptic functions, abelian functions and calculus of variations influenced many generations of mathematicians, and his approach still dominates the teaching of analysis today.

[19]Pavel Sergeevich Aleksandrov, born May 7, 1896 in Bogorodsk, Russia, died November 16, 1982 in Moscow, USSR. In addition to compactness, Aleksandrov introduced local finiteness and laid the foundation of homology theory.

[20]Pavel Samuilovich Urysohn, born February 3, 1898 in Odessa, Ukraine, died August 17, 1924 in Batz-sur-Mer, France. Urysohn is known for the contribution to the theory of dimension.

the original assumption that $B(x_n, \epsilon_n)$ is not contained in any open subset in $\mathcal{U}$. This completes the proof of the claim.

Now we are ready to prove that $X$ is compact. For any open cover $\mathcal{U}$ of $X$, there is $\epsilon > 0$ with the property as stated in the claim above. Starting with any $x_1 \in X$, we try to inductively find a sequence of points $x_1, x_2, \dots,$ such that $d(x_i, x_j) \geq \epsilon$ for $i \neq j$. Specifically, if $x_1, x_2, \dots, x_n$ have been found and $X \neq B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$, then any $x_{n+1} \in X - B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$ satisfies

$$d(x_1, x_{n+1}) \geq \epsilon, \quad d(x_2, x_{n+1}) \geq \epsilon, \quad \dots, \quad d(x_n, x_{n+1}) \geq \epsilon.$$

If the process does not stop, then we get an infinite sequence, and $A = \{x_n : n \in \mathbb{N}\}$ is an infinite subset without limit points (see Exercise 2.5.2). Since the space is assumed to be limit point compact, this cannot happen. Therefore we must have

$$X = B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$$

for some $n$. Since each ball of radius $\epsilon$ is contained in some open subset in $\mathcal{U}$, we conclude that $X$ is covered by $n$ open subsets in $\mathcal{U}$. □

For an open cover $\mathcal{U}$ of a metric space $X$, if a number $\epsilon > 0$ has the property that any ball of radius $\epsilon$ is contained in some $U \in \mathcal{U}$, then we call $\epsilon$ a *Lebesgue*[21] *number* of the open cover. The proof above shows that any open cover of a compact metric space has a Lebesgue number.

Geometrically, for any subset $A$ and $\epsilon > 0$, the subset

$$A^{-\epsilon} = \{x : B(x, \epsilon) \subset A\}$$

is the "$\epsilon$-shrinking" of $A$ defined in Exercise 2.6.17. Then $\epsilon$ is a Lebesgue number of a cover $\mathcal{U}$ if the $\epsilon$-shrinkings $\{U^{-\epsilon} : U \in \mathcal{U}\}$ still form a cover.

**Example 7.6.1.** By default, finite topological spaces are limit point compact.

**Example 7.6.2.** In $\mathbb{R}_{\leftarrow}$, the set of limit points of the single point set $\{a\}$ is $(-\infty, a)$. Therefore any nonempty subset has a limit point, and the topology is limit point compact.

More generally, if any single point subset is not closed, then the closure of $\{x\}$ is strictly bigger than $\{x\}$, so that $\{x\}' \neq \emptyset$. This implies that any nonempty subset has a limit point, and the topology is limit point compact.

**Example 7.6.3.** For a Euclidean ball $B$ of radius $R$, the $\epsilon$-shrinking $B^{-\epsilon}$ is the closed ball of radius $R - \epsilon$ with the same center. Therefore the Lebesgue number for the open cover in Example 7.5.4 are positive numbers $\leq \epsilon - \dfrac{1}{\sqrt{2}}$.

[21]Henri Léon Lebesgue, born June 28, 1875 in Beauvais, France, died July 26, 1941 in Paris, France. His concept of measure revolutionized the integral calculus. He also made major contributions in other areas of mathematics, including topology, potential theory, the Dirichlet problem, the calculus of variations, set theory, the theory of surface area and dimension theory.

Exercise 7.6.1. Suppose a topology is limit point compact. Is a finer topology also limit point compact? What about a coarser topology?

Exercise 7.6.2. Prove that the subsets $A$ in the definition of limit point compactness can be restricted to countable subsets.

Exercise 7.6.3. Prove that the finite complement topology is limit point compact.

Exercise 7.6.4. Prove that any subset of $\mathbb{R}_{\leftarrow}$ is limit point compact.

Exercise 7.6.5. Prove that in the lower limit topology, a subset $A \subset \mathbb{R}$ is limit point compact if and only if it is compact.

Exercise 7.6.6. Prove that if a subset of $\mathbb{R}$ is limit point compact in the lower limit as well as the upper limit topologies, then the subset is finite.

Exercise 7.6.7. Prove that closed subsets of a limit point compact space are still limit point compact.

Exercise 7.6.8. Prove that the union of two limit point compact subsets is still limit point compact.

Exercise 7.6.9. Construct a cover of $\mathbb{R}^2$ by two open subsets, such that the cover has no Lebesgue number.

Exercise 7.6.10. Prove that if a finite open cover has Lebesgue numbers, then the supremum of all the Lebesgue numbers is still a Lebesgue number. Can you drop the finite assumption?

Exercise 7.6.11 (Uniform Continuity Theorem). Let $f\colon X \to Y$ be a continuous map from a compact metric space $X$ to a metric space $Y$. Use the Lebesgue number to prove that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$d(x_0, x_1) < \delta \implies d(f(x_0), f(x_1)) < \epsilon.$$

Exercise 7.6.12 (Contraction Principle). Let $X$ be a compact metric space. A map $f\colon X \to X$ is a *contraction* if $d(f(x), f(y)) \le c\, d(x, y)$ for some constant $c < 1$. Prove that for a contraction $f$, the sequence

$$x,\ f(x),\ f^2(x) = f(f(x)),\ f^3(x) = f(f(f(x))),\ \dots,$$

converges for any $x \in X$. Moreover, the limit $a$ is the unique element satisfying $f(a) = a$ (called the fixed point of $f$).

Exercise 7.6.13. A space is *countably compact* if any countable open cover has a finite subcover.

1. Prove that a space is countably compact if and only if for any closed and nonempty $C_n$ satisfying $C_1 \supset C_2 \supset \cdots$, the intersection $\cap C_n \neq \emptyset$.

2. Let $a_n$ be a non-repetitive sequence in a countably compact space. Let $C_n = \overline{\{a_k : k \geq n\}}$ and $x \in \cap C_n$. Prove that any open subset containing $x$ must contain infinitely many terms in the sequence.
3. Prove that countably compact implies limit point compact.

Exercise 7.6.14. A point $x$ is an *$\omega$-limit point* of a subset $A$ if any open subset containing $x$ must contain infinitely may points in $A$.

1. Prove that a space is countably compact if and only if any infinite subset has an $\omega$-limit point.
2. Prove that for a subset in a Fréchet space, limit points and $\omega$-limit points are the same.

Combined with the Exercise 7.6.13, we find that a Fréchet space is countably compact if and only if it is limit point compact.

Exercise 7.6.15. Prove properties of countably compact spaces.

1. A closed subset of a countably compact space is countably compact.
2. The union of two countably compact subsets is countably compact.
3. The continuous image of a countably compact space is countably compact.

Exercise 7.6.16. A space is *sequentially compact* if any sequence has a convergent subsequence (see Exercise 4.5.16).

1. Prove that sequentially compact implies limit point compact.
2. For Fréchet and first countable spaces (see Example 5.2.8), prove that limit point compact implies sequentially compact.

In particular, a metric space is sequentially compact if and only if it is compact.

Exercise 7.6.17. Prove properties of sequentially compact spaces.

1. A closed subset of a sequentially compact space is sequentially compact.
2. The continuous image of a sequentially compact space is sequentially compact.

Chapter 8

# Surface

## 8.1 Manifold

Consider the subspaces of the plane $\mathbb{R}^2$ in Figure 8.1.1. Note that the spaces on the first row are more "regular" than the ones on the second row. The location of the irregularities in the second row are indicated by the question mark.

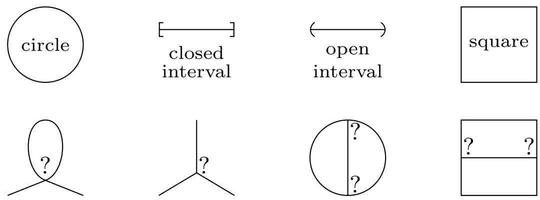



**Figure 8.1.1.** *"regular" and "irregular" spaces*

Mathematically, the observation can be explained as follows. For any point in any space in the first row, there is always an open neighborhood $U$ around the point that is homeomorphic to either $\mathbb{R}$ or $\mathbb{R}_+ = \{x : x \geq 0\}$. Such open neighborhoods do not exist at the questioned points in the spaces on the second row.

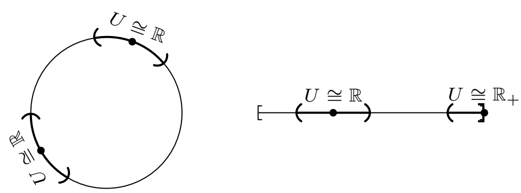



**Figure 8.1.2.** *meaning of "regular"*

Similar case can be made at higher dimensions. See Figure 8.1.3 for some 2-dimensional examples. In general, an *$n$-dimensional (topological) manifold* is a Hausdorff topological space $M$ such that any point has a neighborhood homeomorphic to either the Euclidean space $\mathbb{R}^n$ or the half Euclidean space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n \geq 0\}$.

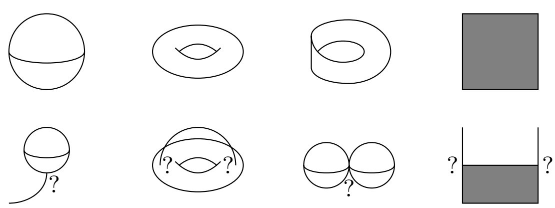



**Figure 8.1.3.** *"regular" and "irregular" spaces at dimension 2*

From the examples in low dimensions, it is easy to see that the points corresponding to $x_n = 0$ for the case $U \cong \mathbb{R}^n_+$ form the so-called *boundary* of the manifold, which is usually denoted as $\partial M$. A manifold $M$ has no boundary if $\partial M = \emptyset$. In other words, we always have $U \cong \mathbb{R}^n$ for manifolds without boundary.

0-dimensional manifolds are isolated points. The circle and the real line are 1-dimensional manifolds without boundary. The closed interval is a 1-dimensional manifold with two end points as the boundary. The sphere and the torus are 2-dimensional manifolds without boundary. The Möbius band and the square are 2-dimensional manifolds with one circle (up to homeomorphism) as the boundary. 2-dimensional manifolds are also called *surfaces*.

A *closed manifold* is a compact manifold without boundary. Note that the use of "closed" here is different from closed subsets in topological spaces. The circle, the sphere, and the torus are all closed manifolds. The unit sphere

$$S^{n-1} = \{(x_1, \dots, x_n) \colon x_1^2 + \cdots + x_n^2 = 1\}$$

in the Euclidean space $\mathbb{R}^n$ is a closed manifold of dimension $n-1$.

The Euclidean space $\mathbb{R}^n$ is a manifold without boundary. But $\mathbb{R}^n$ is not a closed manifold because it is not compact. The unit ball

$$B^n = \{(x_1, \dots, x_n) \colon x_1^2 + \cdots + x_n^2 \leq 1\}$$

is an $n$-dimensional manifold with $\partial B^n = S^{n-1}$.

Exercise 8.1.1. Which alphabets are manifolds?

Exercise 8.1.2. Prove that the real projective space $P^n$ is a manifold by constructing, for any $0 \leq i \leq n$, a homeomorphism between the subset

$$U_i = \{\mathrm{span}\{(x_0, x_1, \dots, x_n)\} \colon x_i \neq 0\} \subset P^n$$

and $\mathbb{R}^n$. Similarly, prove that the complex projective space is also a manifold.

Exercise 8.1.3. Suppose $M$ and $N$ are manifolds and $f \colon \partial M \to \partial N$ is a homeomorphism. Prove that the union $M \cup_f N$ obtained by identifying $x \in \partial M$ with $f(x) \in \partial N$ is a manifold without boundary. We say that $M \cup_f N$ is obtained by gluing the two manifolds along their boundaries.

Exercise 8.1.4. Find a homeomorphism from the Möbius band to itself that reverses the direction of the boundary circle.

Exercise 8.1.5. Prove that if $M$ and $N$ are manifolds, then $M \times N$ is a manifold. Moreover, we have $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.

Exercise 8.1.6. A continuous map $f \colon X \to Y$ is a *local homeomorphism* if any point of $X$ has an open neighborhood $U$, such that $f(U)$ is open in $Y$ and the restriction $f \colon U \to f(U)$ is a homeomorphism.

1. Prove that a local homeomorphism maps open subsets to open subsets.
2. Show that the natural projection $S^n \to P^n$ is a local homeomorphism.

3. Prove that if $f\colon X \to Y$ is a local homeomorphism, then $X$ is a manifold if and only if $Y$ is a manifold.

A fundamental question in topology is the classification of manifolds. Since a space is a manifold if and only if its connected components are manifolds, the problem is really about connected manifolds.

The only 0-dimensional connected manifold is a single point. It can also be shown that the only 1-dimensional connected and closed manifold is the circle. The main purpose of this chapter is to find all connected and closed surfaces.

## 8.2 Surface

The sphere $S^2$ is a closed surface without holes. The torus $T^2$ is a closed surface with one hole. There are also closed surfaces with many holes. The number of holes is called the *genus* of the surface and is denoted by $g(S)$.

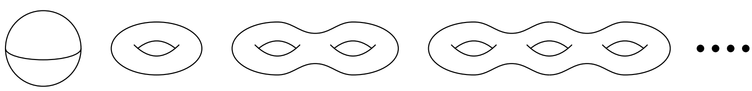


**Figure 8.2.1.** *holes in closed surfaces*

The real projective space $P^2$ is a connected closed surface not listed in Figure 8.2.1. Here is yet another way of constructing $P^2$. Both the Möbius band and the disk have a circle as the boundary. Gluing the two boundary circles together produces a surface without boundary. The result is $P^2$.

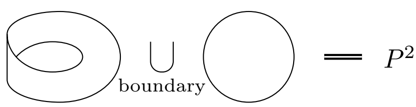



**Figure 8.2.2.** *another way of constructing $P^2$*

The idea for constructing $P^2$ may be applied to other situations. Figure 8.2.3 shows that the Klein bottle $K^2$ can be obtained by gluing two Möbius bands along their boundary circles.

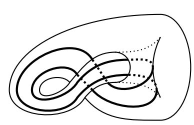


**Figure 8.2.3.** *Cutting along the thick line gives two Möbius bands.*

In general, given two connected closed surfaces $S_1$ and $S_2$, we may delete one disk $\mathring{B}^2$ ($\mathring{B}^2$ is the interior of $B^2$) from each surface, and then glue $S_1 - \mathring{B}^2$ and

$S_2 - \mathring{B}^2$ together along the boundary circles. The result is the *connected sum*

$$S_1 \# S_2 = (S_1 - \mathring{B}^2) \cup_{S^1} (S_2 - \mathring{B}^2)$$

of the two surfaces. For example, since Figure 8.2.2 shows that $P^2 - \mathring{B}^2$ is the Möbius band, Figure 8.2.3 tells us that the Klein bottle $K^2 = P^2 \# P^2$.

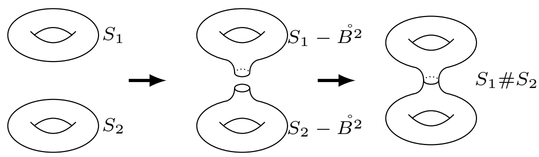



**Figure 8.2.4.** *connected sum*

Strictly speaking, the definition of the connected sum needs to be justified by showing that the surface $S_1 \# S_2$ is, up to homeomorphism, independent of the following choices:

- the size of the disks to be deleted,
- the location of the disks to be deleted,
- the way the two boundary circles are glued together.

To see what happens when the size of the disk is changed, consider two disks $B^2_{\text{small}}$ and $B^2_{\text{big}}$ in a surface $S$ such that $B^2_{\text{small}}$ is contained in the interior of $B^2_{\text{big}}$. Find a disk $B^2_{\text{biggest}}$ in $S$ slightly bigger than $B^2_{\text{big}}$. It is easy to construct (see Exercise 8.2.1) a homeomorphism $\phi$ from $B^2_{\text{biggest}} - \mathring{B}^2_{\text{big}}$ (with boundary $\partial B^2_{\text{biggest}} \cup \partial B^2_{\text{big}}$) to $B^2_{\text{biggest}} - \mathring{B}^2_{\text{small}}$ (with boundary $\partial B^2_{\text{biggest}} \cup \partial B^2_{\text{small}}$), such that the restriction on $\partial B^2_{\text{biggest}}$ is the identity. Then we have a homeomorphism

$$\begin{array}{ccccc}
S - \mathring{B}^2_{\text{big}} & = & (S - \mathring{B}^2_{\text{biggest}}) & \cup & (B^2_{\text{biggest}} - \mathring{B}^2_{\text{big}}) \\
 & & \Big\downarrow id & & \Big\downarrow \phi \\
S - \mathring{B}^2_{\text{big}} & = & (S - \mathring{B}^2_{\text{biggest}}) & \cup & (B^2_{\text{biggest}} - \mathring{B}^2_{\text{small}})
\end{array}$$

Exercise 8.2.1. Let $B^2(r)$ be the disk in $\mathbb{R}^2$ centered at the origin and with radius $r$. Then the triple $(B^2_{\text{biggest}}, B^2_{\text{big}}, B^2_{\text{small}})$ is homeomorphic to $(B^2(r), B^2(1), B^2(s))$, where $r > 1 > s$. Construct a homeomorphism $\psi\colon B^2(r) \to B^2(r)$ such that $\psi(x) = x$ for $x \in \partial B^2(r)$ (a circle of radius $r$) and $\psi(B^2(1)) = B^2(s)$. Then $\psi$ induces a homeomorphism $B^2(r) - \mathring{B}^2(1) \cong B^2(r) - \mathring{B}^2(s)$. Show that this homeomorphism translates into a homeomorphism $B^2_{\text{biggest}} - \mathring{B}^2_{\text{big}} \cong B^2_{\text{biggest}} - \mathring{B}^2_{\text{small}}$ with the desired property.

Now consider what happens when the location of the disk is changed. Since the size does not matter, it is sufficient to consider two points $x$ and $y$ and tiny disks

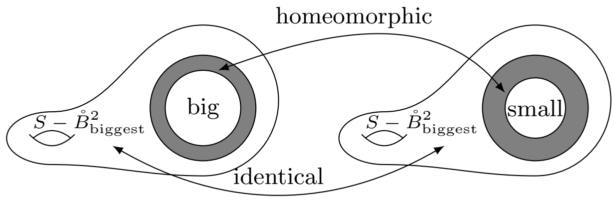



**Figure 8.2.5.** *$S-\mathring{B}^2_{\text{big}}$ and $S-\mathring{B}^2_{\text{small}}$ are homeomorphic.*

$B^2_x$ and $B^2_y$ around them. Find a path connecting $x$ to $y$ without self crossing and then thicken the path a little bit to get a longish region $D \subset S$ that contains both tiny disks $B^2_x$ and $B^2_y$ in the interior. It is easy to construct (see Exercise 8.2.2) a homeomorphism $\phi$ from $D$ to itself, such that the restriction on the boundary $\partial D$ is the identity and $\phi(B^2_x) = B^2_y$. Then we have a homeomorphism

$$\begin{array}{ccccc} S-\mathring{B}^2_x & = & (S-\mathring{D}) & \cup & (D-\mathring{B}^2_x) \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \phi} \\ S-\mathring{B}^2_y & = & (S-\mathring{D}) & \cup & (D-\mathring{B}^2_y) \end{array}$$

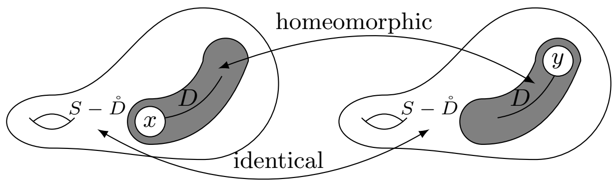



**Figure 8.2.6.** *$S-\mathring{B}^2_x$ and $S-\mathring{B}^2_y$ are homeomorphic.*

Exercise 8.2.2. Let $B^2((a,b),r)$ be the disk in $\mathbb{R}^2$ centered at the point $(a,b)$ and with radius $r$. Then the triple $(D, B^2_x, B^2_y)$ is homeomorphic to $(B^2((0,0),4), B^2((-2,0),1), B^2((2,0),1))$. Construct a homeomorphism $\psi : B^2((0,0),4) \to B^2((0,0),4)$ such that $\psi(x) = x$ for $x \in \partial B^2((0,0),4)$ (a circle of radius 4) and $\psi(B^2((-2,0),1)) = B^2((2,0),1)$. Then $\psi$ translates into a homeomorphism $D \cong D$ with the desired property.

Exercise 8.2.3. Is the connected sum $S_1 \# S_2$ well-defined if $S_1$ or $S_2$ are no longer connected?

Now consider the way the two boundary circles are glued together. If directions are assigned to the two boundary circles, then the gluing map either preserves or reverses the directions. Specifically, let $f\colon \partial(S_1 - \mathring{B}^2) \to \partial(S_2 - \mathring{B}^2)$ be a homeomorphism used in constructing a connected sum $S_1\#_f S_2$. By identifying $\partial(S_1 - \mathring{B}^2)$ with $S^1 = \{z\colon z \in \mathbb{C}, |z| = 1\}$, we may introduce a homeomorphism $\rho\colon \partial(S_1 - \mathring{B}^2) \to \partial(S_1 - \mathring{B}^2)$ corresponding to the complex conjugation map $z \to \bar{z}\colon S^1 \to S^1$. Then $f$ and $f\rho$ identify the boundaries in reverse directions, so that the other connected sum is $S_1\#_{f\rho}S_2$.

If the homeomorphism $\rho$ can be extended from the boundary to a homeomorphism $\phi\colon S_1-\mathring{B}^2 \to S_1-\mathring{B}^2$ of the whole, then we may construct a homeomorphism

$$\begin{array}{ccccc} S_1\#_{f\rho}S_2 & = & (S_1 - \mathring{B}^2) & \cup_{f\rho} & (S_2 - \mathring{B}^2) \\ & & \phi\big\downarrow & & \big\downarrow id \\ S_1\#_f S_2 & = & (S_1 - \mathring{B}^2) & \cup_f & (S_2 - \mathring{B}^2) \end{array}$$

Therefore different ways of gluing the boundary circles together produce homeomorphic surfaces. For the cases $S_1 = T^2$ and $P^2$, such an extension $\phi$ can be constructed explicitly (see Exercises 8.1.4 and 8.2.4). This is already sufficient for us to prove the classification of surfaces. For general surfaces, the existence of $\phi$ can be proved by making use of the classification Theorem 8.6.1 (also see Exercise 8.7.7).

Exercise 8.2.4. Let $M$ be obtained by deleting a disk from the torus $T^2$. Describe a homeomorphism from $M$ to itself such that the restriction on the boundary reverses the boundary circle.

Connected sum can also be defined for manifolds of other dimensions. By essentially the same reason, the connected sum of two *connected* manifolds is independent of the choice of the disks. However, the direction reversing (more precisely, orientation reversing) homeomorphism $\rho$ of the boundary sphere may not always be extended to a homeomorphism $\phi$ in general. Therefore in the definition of the connected sum at higher dimensions, the orientation of the boundary spheres must be specified.

The connected sum construction can be used repeatedly. For example, the surface in Figure 8.2.7 is the connected sum of $g$ copies of the torus, denote by $gT^2$. The connected sum of $g$ copies of $P^2$ is also a closed surface, denoted by $gP^2$.

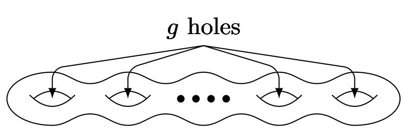



**Figure 8.2.7.** *closed surface with g holes*

**Lemma 8.2.1.** *The connected sum of connected and closed surfaces has the following properties:*

1. *commutative:* $S_1 \# S_2 = S_2 \# S_1$.
2. *associative:* $(S_1 \# S_2) \# S_3 = S_1 \# (S_2 \# S_3)$.
3. *identity:* $S \# S^2 = S$.

The connected sums of copies of the sphere $S^2$, the torus $T^2$, and the projective space $P^2$ produce many connected and closed surfaces. By Lemma 8.2.1 and the fact that $T^2 \# P^2 = P^2 \# P^2 \# P^2$ (proved later in Lemma 8.5.3), all the surfaces thus obtained are:

1. sphere $S^2$.
2. connected sums of tori: $gT^2$, $g \in \mathbb{N}$.
3. connected sums of projective spaces: $gP^2$, $g \in \mathbb{N}$.

In Theorem 8.6.1, we will show that this is the complete list of connected and closed surfaces.

Exercise 8.2.5. Prove that the construction of $P^2$ in Figure 8.2.2 is consistent with the constructions in Section 6.3.

Exercise 8.2.6. By choosing special disk, special glueing, and making use of the fact $S^2 - \mathring{B}^2 \cong B^2$, prove the properties of the connected sum in Lemma 8.2.1.

Exercise 8.2.7. What do you get by shrinking the boundary circle of the Möbius band to a point?

Exercise 8.2.8. The "self-connected sum" can be constructed as follows: For any (connected) surface $S$, remove two disjoint disks from $S$ and then glue the two boundary circles together. What do you get? The answer may depend on one of the two ways the circles are glued together.

## 8.3 Simplicial Surface

Our current definition of surfaces is not easy to work with. To classify connected and closed surfaces, a more constructive way of describing them is needed. Thus we take the simplicial approach and ask how can each point $x$ in a simplicial complex $K$ have a neighborhood homeomorphic to $\mathbb{R}^2$? Note that $K$ must be 2-dimensional, meaning that $K$ contains only vertices (0-simplices), edges (1-simplices), and faces (2-simplices). Therefore there are three possibilities for the the location of the point $x$.

Suppose $x$ is in the interior of a face. Then the whole interior of the face is a neighborhood of $x$ homeomorphic to $\mathbb{R}^2$. Thus the manifold condition is satisfied for such $x$.

Suppose $x$ is in the interior of an edge. Then the neighborhood of $x$ in $K$ involves all the faces with the edge as part of the boundary. Figure 8.3.2 suggests the following condition for the existence of a neighborhood homeomorphic to $\mathbb{R}^2$.

**A.** *Each edge is shared by exactly two faces.*

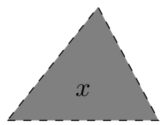



**Figure 8.3.1.** *the neighborhood of a point inside a face*

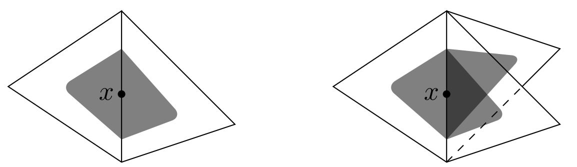



**Figure 8.3.2.** *the neighborhood of a point inside an edge*

Finally, suppose $x$ is a vertex. Then the neighborhood of $x$ involves all the edges and faces containing this vertex. Since the condition **A** above requires that every edge belongs to some face, we simply need to consider all faces with $x$ as a vertex. Let $\sigma_1$ be one such face. Then $x$ belongs to two edges $\epsilon_1, \epsilon_2$ of $\sigma_1$. By the condition **A**, the edge $\epsilon_2$ also belongs to exactly one another face $\sigma_2$. Then $x$ belongs to two edges $\epsilon_2, \epsilon_3$ of $\sigma_2$. Now the condition **A** says again that $\epsilon_2$ also belongs to exactly one another face $\sigma_3$. Keep going, we get faces $\sigma_1, \sigma_2, \dots, \sigma_n$, such that $\sigma_i$ and $\sigma_{i+1}$ share one edge $\epsilon_{i+1}$ that contains $x$. Since all our simplicial complexes are finite, eventually some $\sigma_n$ and an existing $\sigma_j$ will share one edge $\epsilon_{n+1}$ that contains $x$. By condition **A** and the fact that $\epsilon_{n+1}$ contains $x$, we find this $\sigma_j$ must be $\sigma_1$, and $\epsilon_{n+1} = \epsilon_1$. The situation is described on the left of Figure 8.3.3.

Now we claim that the cycle of faces $\sigma_1, \sigma_2, \dots, \sigma_n$ is all the faces that contains $x$. If not, then we start from a face containing $x$ that does not belong to the cycle. The condition **A** then implies that we may construct another cycle, and this new cycle does not contain any faces in the cycle $\sigma_1, \sigma_2, \dots, \sigma_n$. The situation is described on the right of Figure 8.3.3. Then we see that the neighborhood of $x$ cannot be homeomorphic to $\mathbb{R}^2$. Therefore we conclude the following condition for the existence of a neighborhood of a vertex homeomorphic to $\mathbb{R}^2$.

**B.** *For each vertex, all the faces containing the vertex point can be cyclically ordered as $\sigma_1, \sigma_2, \dots, \sigma_n$, ($\sigma_{n+1} = \sigma_1$), such that $\sigma_i$ and $\sigma_{i+1}$ share one edge that contains the vertex.*

In subsequent sections, the following is our working definition of surfaces: A closed surface is a finite 2-dimensional simplicial complex satisfying conditions **A** and **B**. Strictly speaking, it needs to be shown that this definition is equivalent to

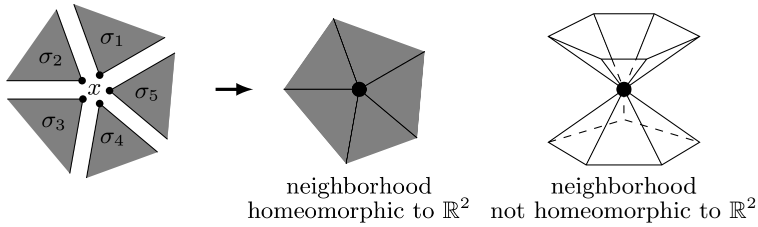



**Figure 8.3.3.** *the neighborhood of a vertex*

the definition of a surface as a compact 2-dimensional manifold without boundary. This was done by Radó[22] in 1925. The proof is omitted here.

Exercise 8.3.1. Prove that finite and connected 1-dimensional simplicial manifolds must be circles and closed intervals.

Exercise 8.3.2. What are the conditions satisfied by simplicial surfaces *with boundary*?

Exercise 8.3.3. What are the conditions for a finite 3-dimensional simplicial complex to be a manifold without boundary?

## 8.4 Planar Diagram

A CW-structure of the sphere $S^2$ consists of two 0-cells $v$ and $w$, one 1-cell $a$, and one 2-cell $B^2$. The boundary of the 2-cell is divided into two edges, and the sphere is obtained by identifying the two edges.

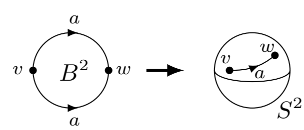



**Figure 8.4.1.** *a planar diagram of the sphere*

A CW-structure of the projective space $P^2$ consists of one 0-cell $v$, one 1-cell $a$, and one 2-cell $B^2$. The boundary of the 2-cell is divided into two edges, and the projective space is obtained by identifying the two edges as in Figure 8.4.2, which is the same as identifying the antipodal points. Also note that the identification of the two edges forces the two vertices to become the same one.

A CW-structure of the torus $T^2$ consists of one 0-cell $v$, two 1-cells $a$ and $b$, and one 2-cell $B^2$. The boundary of the 2-cell is divided into four edges, and the torus is obtained by identifying these edges as indicated in Figure 8.4.3. Note that

[22]Tibor Radó, born June 2, 1895 in Budapest, Hungary, died December 12, 1965 in Daytona Beach, Florida, USA. Radó made contributions to many fields of mathematics, including analysis, geometry and theoretical computer science.

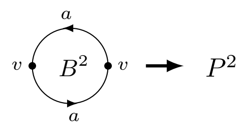



**Figure 8.4.2.** *a planar diagram of the projective space*

the identification of the edges forces the four vertices at the corners of the square to become the same one.

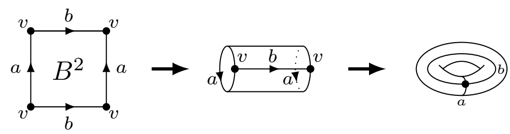



**Figure 8.4.3.** *a planar diagram of the torus*

A CW-structure of $K^2$ consists of one 0-cell $v$, two 1-cells $a$ and $b$, and one 2-cell $B^2$. The boundary of the 2-cell is divided into four edges, and the Klein bottle is obtained by identifying the edges as in Figure 8.4.4. Note that the identification of the edges forces the four vertices to become the same one.

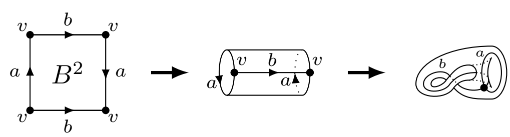



**Figure 8.4.4.** *a planar diagram of the Klein bottle*

Exercise 8.4.1. What is the surface obtained by gluing edges of a square as in Figure 8.4.5?

All the examples shown above fit into the following description.

1. The boundary of a disc is divided into several edges.

2. The edges are grouped into pairs and assigned arrows.

3. The (connected and closed) surface is obtained by identifying all the pairs of edges in the way indicated by the assigned arrows.

The disk with indicated pairs and arrows on the boundary is called a *planar diagram* of the surface.

The identification of the edge pairs in a planar diagram forces some vertices in the diagram to be identified. For example, in Figure 8.4.6, six vertices are identified

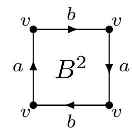



**Figure 8.4.5.** *What surface do you get?*

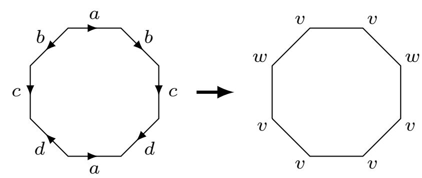



**Figure 8.4.6.** *The number of vertices must be two.*

as $v$, and the remaining two are identified as $w$. With the identification of vertices understood, the planar diagram gives a CW-structure of the surface.

The following result reduces the study of surfaces to planar diagrams.

**Lemma 8.4.1.** *Any connected and closed surface is given by a planar diagram.*

*Proof.* Consider a surface $S$ as a 2-dimensional simplicial complex satisfying conditions **A** and **B**. We will try to inductively arrange all faces into a sequence $\sigma_1, \sigma_2, \dots, \sigma_n$, such that $\sigma_i$ intersects the union $\cup_{j<i}\sigma_j$ of all the preceeding faces along some edges.

Suppose we have found $\sigma_1, \sigma_2, \dots, \sigma_k$ to satisfy the desired condition. Let $\tau_1, \tau_2, \dots, \tau_l$ be the remaining faces. Since $S$ is connected, there are some $\sigma_i$ and $\tau_j$ sharing a vertex $v$. By condition **B**, all the faces with $v$ as a vertex can be arranged in the cyclic way. Since such faces include some $\sigma_i$ and some $\tau_j$, and the arrangement is cyclic, there must be some $\sigma_{i'}$ and some $\tau_{j'}$ adjacent in the arrangement, so that the two faces share one edge. In particular, the intersection $\tau_{j'} \cap [\cup_{i\leq k}\sigma_i]$ contains at least one edge. Therefore if we take $\sigma_{k+1} = \tau_{j'}$, then $\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}$ still satisfy the desired condition.

Having arranged all the faces into a sequence $\sigma_1, \sigma_2, \dots, \sigma_n$ with the desired property, we glue the faces successively in the order of the sequence, and each time *along exactly one edge*. For example, in the triangulation of the torus in Figure 8.4.7, the intersections $\sigma_1 \cap \sigma_2$, $(\sigma_1 \cup \sigma_2) \cap \sigma_3$, $\dots$, $(\sigma_1 \cup \cdots \cup \sigma_4) \cap \sigma_5$ are all just single edges, and the each gluing happens only along the single edge. However, $(\sigma_1 \cup \cdots \cup \sigma_5) \cap \sigma_6$ consists of two edges $\sigma_5 \cap \sigma_6$ and $\sigma_1 \cap \sigma_6$. In constructing the planar diagram, $\sigma_6$ is glued to $\sigma_5 \subset \sigma_1 \cup \cdots \cup \sigma_5$ and not glued to $\sigma_1$. Similar choices are made when $\sigma_8$, $\sigma_{10}$, $\sigma_{11}$, $\sigma_{12}$, $\sigma_{13}$, $\sigma_{14}$ are glued to the previous faces.

Suppose after the $k$-th step, we get a disk $B(k)$. Then all the edges in the interior of the disk have been identified. By condition **A**, the common edge(s) between $\sigma_{k+1}$ and $\cup_{j\leq k}\sigma_j$ must lie on the boundary of $B(k)$. Therefore by gluing

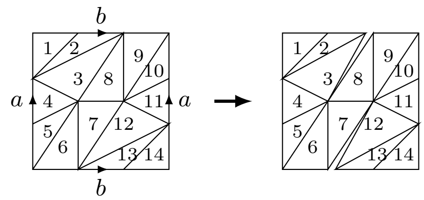



**Figure 8.4.7.** *from triangulation to planar diagram*

$\sigma_{k+1}$ to $B(k)$ along only one such common edge, the result $B(k+1)$ is still a disk. At the end, we get a disk $B(n)$.

Of course, the edges on the boundary of $B(n)$ must still be glued together to form $S$. Condition **A** says that each boundary edge is identified with exactly one other boundary edge. Moreover, the identification of the vertices forced by the identification of the edge pairs satisfies the condition **B**, and any additional identification of the vertices will violate the condition **B**. This makes $B(n)$ into a planar diagram. $\square$

Instead of drawing pictures all the time, there is a simple way of denoting planar diagrams. First give names (such as $a$, $b$, ...) to the edge pairs. Then start from some vertex and traverse along the boundary (in either direction). Along the "trip" a sequence of names are encountered. A word is then made by using "name" when the direction of the trip is the same as the assigned arrow and by using "name$^{-1}$" when the direction of the trip is opposite to the assigned arrow. For example, the planer diagrams for $S^2$, $P^2$, $T^2$, $K^2$ given at the beginning of this section can be denoted as $aa^{-1}$, $aa$, $aba^{-1}b^{-1}$, $abab^{-1}$, respectively. The planar diagram in Figure 8.4.6 can be denoted as $abcda^{-1}dc^{-1}b^{-1}$, or $ad^{-1}c^{-1}b^{-1}a^{-1}bcd^{-1}$, or $b^{-1}a^{-1}bcd^{-1}ad^{-1}c^{-1}$.

## 8.5 Cut and Paste

The following result tells us that, if two surfaces are given by planar diagrams denoted by words $w_1$ and $w_2$, then their connected sum is given by a planar diagram denoted by the word $w_1w_2$.

**Lemma 8.5.1.** *Given planar diagrams of two surfaces, the planar diagram for the connected sum is given as in Figure 8.5.1.*

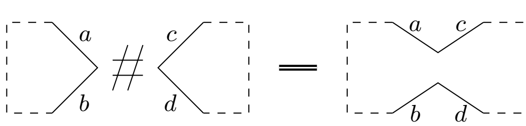



**Figure 8.5.1.** *planar diagram of connected sum*

*Proof.* The proof is illustrated in Figure 8.5.2. The disks used for constructing the connected sum can be chosen anywhere. In particular, they can be chosen near vertices, as indicated by the shaded regions in the figure. The boundaries of both disks are denoted by the same name $e$ because they are going to be identified in the connected sum construction. After identifying $e$, we get the planar diagram for the connected sum. □

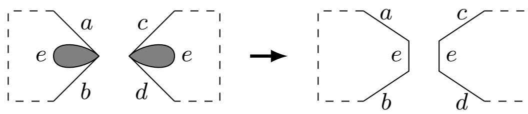



**Figure 8.5.2.** *construct the planar diagram of connected sum*

As a consequence of the lemma, we have the *standard planar diagrams* in Figure 8.5.3.

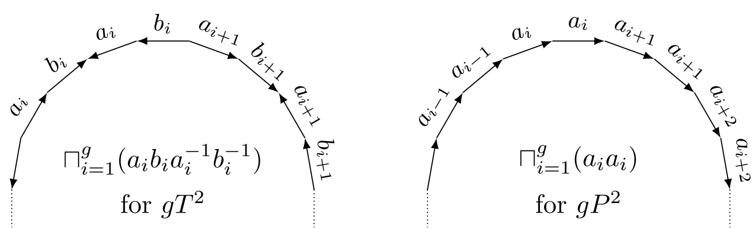



**Figure 8.5.3.** *standard planar diagrams*

A surface may be given by many different planar diagrams. We may use the *cut and paste* technique to show that two different planar diagrams induce the same surface. The technique is critical to the proof of the classification theorem. Here we practice the technique by simple examples.

**Lemma 8.5.2.** *The planar diagrams $aabb$ and $abab^{-1}$ give the same surfaces.*

By Lemma 8.5.1 and the fact that $aa$ gives the projective space $P^2$, $aabb$ gives the connected sum $P^2\#P^2$. From Figure 6.1.11, $abab^{-1}$ gives the Klein bottle. Thus the lemma basically says $K^2 = P^2\#P^2$. The fact was also illustrated in Figure 8.2.3.

*Proof.* Starting with the planar diagram $abab^{-1}$, Figure 8.5.4 shows how to manipulate the gluing process and get the planar diagram $aabb$ at the end. Each step of the manipulation is taken to make sure that the surfaces represented by the diagrams are the same.

Here we explain in detail what is really going on. The first arrow means that the square may be obtained by gluing two triangles along the edge pair $c$. In this

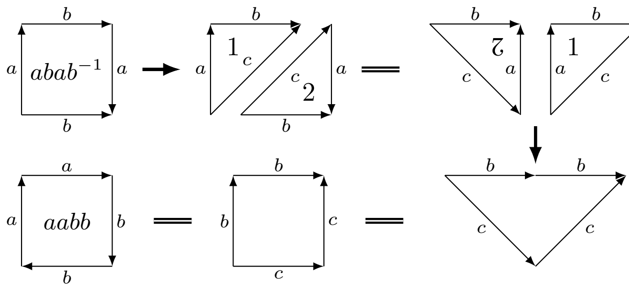



**Figure 8.5.4.** *$abab^{-1}$ is equivalent to $aabb$.*

way, the surface is obtained by gluing two triangles together along three edge pairs $a$, $b$, and $c$.

The first equality is simply another way of drawing the pair of triangles. The first triangle is kept as is and the second triangle is flipped upside down.

In the second arrow, the edge pair $a$ is glued first and the other two pairs are left alone.

The second equality means that the triangle is homeomorphic to the square, with edge pairs $b$ and $c$ unchanged.

The last equality simply renames the edge pairs $b$ to $a$ and $c$ to $b^{-1}$. This does not affect the identification of the boundary edges. □

**Lemma 8.5.3.** $T^2\#P^2 = P^2\#P^2\#P^2$.

*Proof.* The surface $T^2\#P^2$ is given by the planar diagram $aba^{-1}b^{-1}cc$. By Lemma 8.5.2, the surface $P^2\#P^2\#P^2 = K^2\#P^2$ and is given by the planar diagram $abab^{-1}cc$. The proof that the two planar diagrams produce the same surface is given by Figure 8.5.5. □

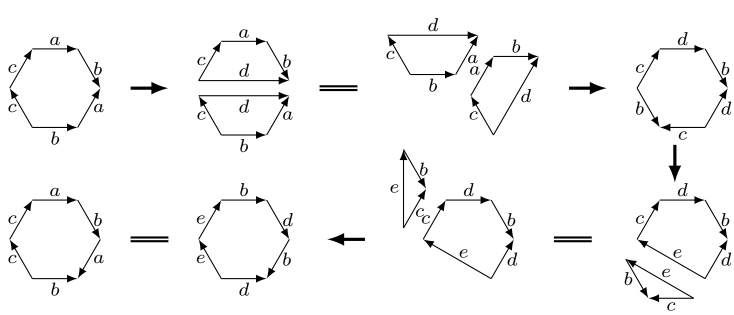


**Figure 8.5.5.** *$aba^{-1}b^{-1}cc$ is equivalent to $abab^{-1}cc$.*

Exercise 8.5.1. Figure 8.5.6 shows two Möbius bands after identifying the pairs $a$ and $b$. Explain that the further identification of the pairs $c$ and $d$ means gluing the two Möbius bands together along their boundary circles. Then explain that the result is the same as the Klein bottle as constructed in Figure 8.4.4.

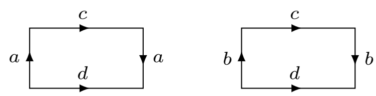



**Figure 8.5.6.** *glueing two Möbius bands together along their boundary circles*

Exercise 8.5.2. There are essentially six different square diagrams. Find these square diagrams and identify the surfaces they represent.

Exercise 8.5.3. There are many triangle diagrams with edges labeled $a$, $b$, $c$, some of which are shown in Figure 8.5.7. Gluing any two triangle diagrams (with the corresponding identifications among six vertices) produces a surface. Identify all such surfaces.

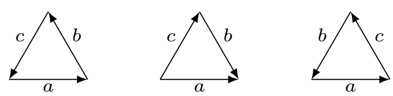



**Figure 8.5.7.** *triangle diagrams*

## 8.6 Classification of Surface

The following classification theorem says that the list of surfaces in Section 8.2 is complete. The theorem was first established by Dehn[23] and Heegaard[24] in 1907.

**Theorem 8.6.1.** *Every connected closed surface is homeomorphic to a sphere, or a connected sum of several tori, or a connected sum of several projective spaces.*

*Proof.* By Lemma 8.4.1, a connected closed surface is given by a planar diagram. Edge pairs in a planar diagram are either *opposing* or *twisted*, as indicated in Figure 8.6.1. In terms of the words, an opposing pair appears as $\cdots a \cdots a^{-1} \cdots$, and a twisted pair appears as $\cdots a \cdots a \cdots$.

[23]Max Dehn, born November 13, 1878 in Hamburg, Germany, died June 27, 1952 in Black Mountain, North Carolina, USA. Dehn is famous for his work in geometry, topology and geometric group theory.

[24]Poul Heegaard, born November 2, 1871 in Copenhagen, Denmark, died February 7, 1948 in Oslo, Norway. Heegaard splitting and Heegaard diagram are still the important tools for studying 3-dimensional manifolds.

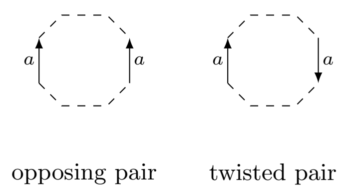



**Figure 8.6.1.** *two possible types of edge pairs*

Given any planar diagram, we first simplify the diagram by combining identical strings (see Figure 8.6.2) and eliminating adjacent opposing pairs (see Figure 8.6.3).

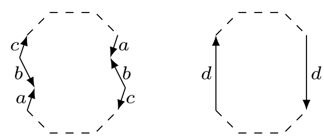



**Figure 8.6.2.** *combining identical strings*

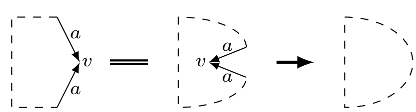



**Figure 8.6.3.** *eliminating adjacent opposing pairs*

If the two simplifications reduce the planar diagram to $aa^{-1}$ (see Figure 8.4.1), then all edge pairs are eliminated and the surface is $S^2$. If the resulting diagram is more sophisticated than $aa^{-1}$, then we will further reduce the number of vertices to one.

Suppose there are more than one vertices. Then there is one edge $a$ for which the two ends are different vertices $v$ and $w$. Let the end $v$ be the beginning of an edge $b$. We cannot have $b = a$ because this implies $w = v$. We cannot have $b = a^{-1}$ because this gives an adjacent opposing pair, which should have already been eliminated. Therefore $a$, $b$ is not an edge pair in the planner diagram, and the planar diagram must be either the one in Figure 8.6.4 or a similar in which the pair $b$ is opposing instead of twisted. Then the cut and paste process in Figure 8.6.4 shows that the number of times $v$ appears can be reduced by one (at the cost that the number of times $w$ appears is increased by one). The process may be repeated until the number of times $v$ appears is reduced to exactly once. Then on the two sides of $v$ must be an adjacent opposing pair, as in Figure 8.6.3. By eliminating the adjacent opposing pair, $v$ can be completely eliminated. Repeating the process will eventually reduce the number of vertices to one.

After the preliminary simplifications, we now try to simplify the diagram to become a product of $aa$ and $aba^{-1}b^{-1}$. Since $aa$ comes from a twisted pair and $aba^{-1}b^{-1}$ comes from (two) opposing pairs, this involves two steps. For the simpler

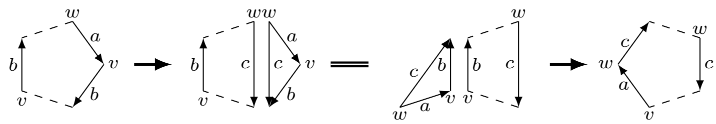


**Figure 8.6.4.** *reduce the number of times a vertex appears*

case of twisted pairs, the cut and paste process in Figure 8.6.5 shows how a twisted pair $\cdots a \cdots a \cdots$ can be brought together to form $\cdots aa \cdots$. (The pair $b$ in the final step may be renamed $a$.) We do this for all twisted pairs so that all the twisted pairs are adjacent.

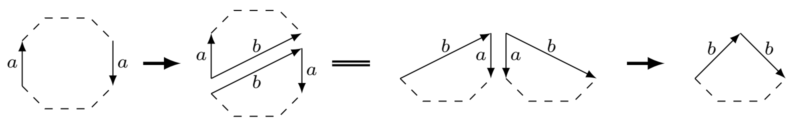


**Figure 8.6.5.** *bring twisted pairs together*

For an opposing pair $\cdots a \cdots a^{-1} \cdots$, we need to first find another opposing pair $b$ located as $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$ (see Figure 8.6.6). Then we need to bring the two pairs together to form $\cdots aba^{-1}b^{-1} \cdots$.

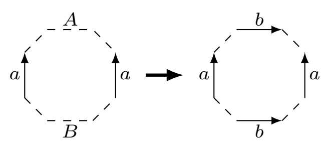


**Figure 8.6.6.** *the existence of a pair of opposing pairs*

Since all adjacent opposing pairs have been eliminated, the pair $a$ must separate the boundary into two nonempty parts $A$ and $B$, as shown on the left of Figure 8.6.6. Since there is only one vertex, some vertex in $A$ must be identified with some vertex in $B$. In other words, there is a pair $b$ with one part in $A$ and the other part in $B$. In particular, the two parts of $b$ are separated. Since all twisted pairs have been brought together, the pair $b$ must be an opposing one. So we get the situation on the right of Figure 8.6.6. Then the cut and paste process in Figure 8.6.7 further brings the pairs $a$ and $b$ together to form $\cdots aba^{-1}b^{-1} \cdots$.

Now the planar diagram is reduced to a product of $aa$ and $aba^{-1}b^{-1}$. By Lemma 8.5.1, the surface is a connected sum of copies of $P^2$ and $T^2$. The classification theorem then follows from Lemma 8.2.1 and Lemma 8.5.3. □

Exercise 8.6.1. Prove that $T^2 \# K^2 = K^2 \# K^2$.

Exercise 8.6.2. According to the classification theorem, $mT^2 \# nP^2$ must be either $kT^2$ or $kP^2$. Express $k$ in terms of $m$ and $n$.

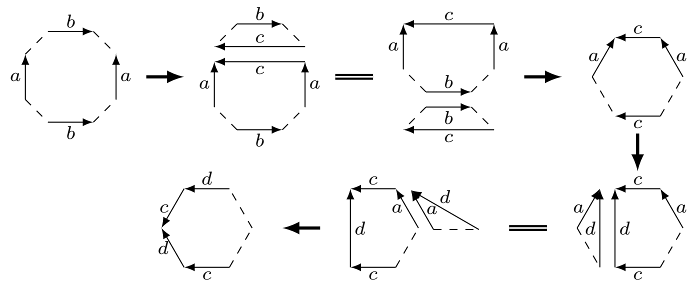


**Figure 8.6.7.** *bring a pair of opposing pairs together*

Exercise 8.6.3. What are the surfaces given by the following planar diagrams?

1. $abab$.
2. $hongkongh^{-1}k^{-1}$.
3. $abcdec^{-1}da^{-1}b^{-1}e^{-1}$.
4. $ae^{-1}a^{-1}bdb^{-1}ced^{-1}c^{-1}$.
5. $abc^{-1}d^{-1}ef^{-1}fe^{-1}dcb^{-1}a^{-1}$.
6. $abcdbeafgd^{-1}g^{-1}hcife^{-1}ih^{-1}$.

## 8.7 Recognition of Surface

Theorem 8.6.1 lists all closed surfaces. However, two important problems remain:

1. Is there any duplication in the list? For example, it appears that $2T^2 \neq 4T^2$. However, this needs to be proved.

2. How to recognize a closed surface in an efficient way? The problem is illustrated by Exercise 8.6.3 and the problem in Figure 8.7.1.

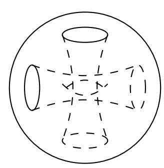


**Figure 8.7.1.** *What is this surface?*

The answer is given by two invariants: Euler number and orientability. The Euler number can be computed from the planar diagram. For example, consider the surface given by the planar diagram $abcdec^{-1}da^{-1}b^{-1}e^{-1}$ in Figure 8.7.2. First the diagram implies that there is only one 0-cell $v$. Since there are five identification pairs of edges, the number of 1-cells is 5. Moreover, there is always one 2-cell in any planar diagram. Therefore the Euler number $\chi(S) = 1 - 5 + 1 = -3$. The following result implies the surface has to be $5P^2$.

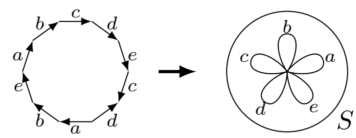



**Figure 8.7.2.** *compute the Euler number*

**Lemma 8.7.1.** *The Euler numbers of connected closed surfaces are* $\chi(S^2) = 2$, $\chi(gT^2) = 2 - 2g$, $\chi(gP^2) = 2 - g$. *In particular, for connected closed surfaces we have*

1. $\chi(S) = 2 \implies S = S^2$.
2. $\chi(S)$ *is odd* $\implies S = (2 - \chi)P^2$.

Exercise 8.7.1. Prove that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$. Then use the formula and the classification theorem to prove Lemma 8.7.1.

Exercise 8.7.2. Let $v$, $e$, $f$ be the numbers of vertices, edges, and faces in a triangulation of a closed surface $S$.

1. Prove that $2e = 3f$.
2. Use the Euler formula and the first part to express $e$ and $f$ in terms of $v$.
3. In a simplicial complex, there is at most one edge between any two vertices. Use the fact to prove that $v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(S)})$.

Let

$$V(S) = \left\lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi(S)}) \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer $\geq x$. Exercise 8.7.2 shows that $V(S)$ is a lower bound for the number of vertices in a triangulation of the surface. Jungerman and Ringel proved[25] that, with the exceptions of $2T^2$, $K^2$ and $3P^2$, the lower bound can be achieved. In the three exceptional cases, the minimal number is $V(S) + 1$.

Complementary to the number $V(S)$ is the *Heawood*[26] *number*

$$H(S) = \left\lfloor \frac{1}{2}(7 + \sqrt{49 - 24\chi(S)}) \right\rfloor,$$

where $\lfloor x \rfloor$ is the biggest integer $\leq x$. In 1890, Heawood proved that, if $S$ is not a sphere, then $H(S)$ number of colors is sufficient to color any map on $S$. He also proved $H(T^2) = 7$ number of colors is necessary for $T^2$ and conjectured that $H(S)$ number of colors is always necessary for any $S$. It is also not difficult to show that

[25] Ringel proved the nonorientable case in 1955. Jungerman and Ringel proved the orientable case in 1980.

[26] Percy John Heawood, born September 8, 1861 in Newport, Shropshire, England, died January 24, 1955 in Durham, England. Heawood spent 60 years of his life on the four color problem.

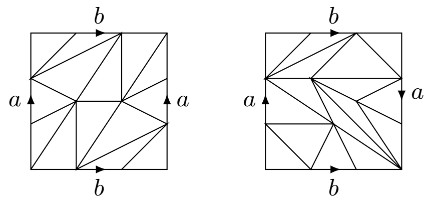



**Figure 8.7.3.** *most efficient triangulations of* $T^2$ *and* $K^2$

the conjecture is true for $P^2$. In 1968, Ringel and Youngs proved the conjecture in case $\chi < 0$. In 1976, with the help of computer, Appel and Haken solved the conjecture for $S^2$ (the famous *four color problem*). This leaves the case of Klein bottle. Franklin showed in 1934 that only *six* colors are needed for maps on $K^2$, while $H(K^2) = 7$. Therefore the Heawood conjecture is true for all surfaces except the Klein bottle.

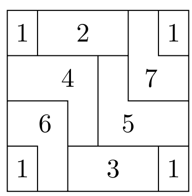



**Figure 8.7.4.** *a torus map that requires seven colors*

Note that $gT^2$ and $2gP^2$ have the same Euler number, although they are supposed to be different surfaces. (For $g = 1$, we are comparing $T^2$ and $K^2$.) To distinguish the two surfaces, therefore, another property/invariant is needed in addition to the Euler number. Such a property/invariant is the *orientability.*

The plane $\mathbb{R}^2$ has two orientations, represented by a circle indicated with one of two possible directions: clockwise or counterclockwise. Since a surface is locally homeomorphic to the plane, there are two possible orientations at any point of the surface. A choice of the orientation at a point $x$ will be indicated by a small circle around the point with a choice of direction and will be denoted as $o_x$.

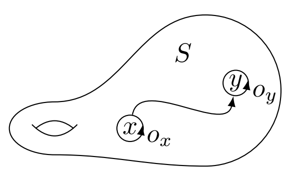



**Figure 8.7.5.** *moving orientation from* $x$ *to* $y$ *along a path*

An orientation $o_x$ at a point $x$ determines "compatible" orientations $o_y$ for points $y$ near $x$. Along any path from $x$ to $y$, the orientation $o_x$ can be "moved" in compatible way to an orientation $o_y$. For a connected surface, we may fix a point $x$, fix an orientation $o_x$, and then move $o_x$ to any other point along various paths starting from $x$. There are two possible outcomes:

1. No matter which path is chosen from $x$ to $y$, moving $o_x$ along the path always gives us the same orientation at $y$.

2. There are two paths from $x$ to $y$, such that moving $o_x$ along the two paths gives us different orientations at $y$.

In the first case, there is a compatible system of orientations at all points of the surface. Such a system is a (global) *orientation* of the surface and the surface is said to be *orientable*. For example, the sphere and the torus are orientable. In the second case, there is no compatible system of orientations at all points of the surface. Such surfaces are *nonorientable*.

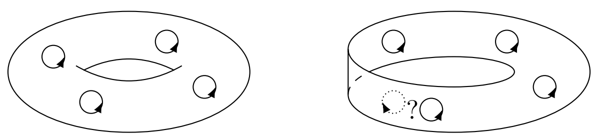



**Figure 8.7.6.** *orientable and nonorientable surfaces*

The nonorientable situation is equivalent to the following: There is a path from $x$ to $x$ (called a loop at $x$), such that moving $o_x$ along the loop and back to $x$ produces an orientation different from $o_x$. It can be further assumed that the loop has no self-intersection because any loop can be broken into several loops with no self-intersection. The region swept by the little circle indicating $o_x$ as we move along the loop is a strip surrounding the loop. The fact that $o_x$ comes back to be different from $o_x$ means exactly that the strip is a Möbius band. This leads to the following characterization of nonorientable surfaces.

**Lemma 8.7.2.** *A surface is nonorientable if and only if it contains the Möbius band.*

By the lemma, the projective space and the Klein bottle are nonorientable.

Exercise 8.7.3. For a surface $S$ embedded in $\mathbb{R}^3$, a *normal vector field* means that any $x \in S$ is continuously assigned a nonzero vector $n_x$ orthogonal to the tangent plane $T_xS$ of the surface at the point.

1. Identifying the sphere $S^2$ as the collection of all unit vectors $\{x \in \mathbb{R}^3 : ||x|| = 1\}$, show that $n_x = x$ is a normal vector field.
2. Prove that for any normal vector field, an orientation for $S$ may be constructed by the "right hand rule" with regard to the normal directions. In particular, the existence of a normal vector field implies the orientability.
3. Explain why the torus is orientable.
4. Explain why the surface in Figure 8.7.1 is orientable.

Exercise 8.7.4. Prove that the connected sum of two surfaces is orientable if and only if each surface is orientable.

Exercise 8.7.5. Use Lemma 8.7.2, Exercises 8.7.3 and 8.7.4 to determine the orientability of all connected closed surfaces.

The orientability of a surface can be determined from its planar diagram. Fixing an orientation for the interior of the planar diagram, the problem is then the compatibility of the local orientations when the boundary edges are glued together. Figure 8.7.7 shows that the incompatibility happens if and only if there are twisted pairs. In terms of the word that denotes the planar diagram, the surface is orientable if and only if the following is satisfied: If $a$ appears in the word, then $a^{-1}$ must also appear. For example, $abcdec^{-1}da^{-1}b^{-1}e^{-1}$ represents a nonorientable surface because $d$ appears but $d^{-1}$ does not appear. Applying this criterion to the standard planar diagrams determines the oriantability of connected closed surfaces.

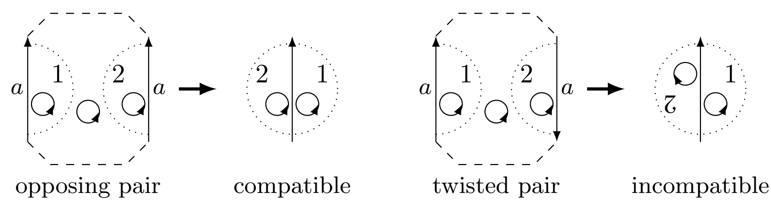



**Figure 8.7.7.** *compatibility of orientations across an edge pair*

**Lemma 8.7.3.** *$S^2$ and $gT^2$ are orientable. $gP^2$ is not orientable.*

Combining Lemma 8.7.1 and Lemma 8.7.3, we have the following method for determining whether two connected closed surfaces are the same.

**Theorem 8.7.4.** *Two connected closed surfaces are homeomorphic if and only if they have the same Euler number and the same orientability.*

Exercise 8.7.6. What are the surfaces given by the following planar diagrams?

1. $hongkongh^{-1}k^{-1}$.
2. $abcdeac^{-1}edb^{-1}$.
3. $abc^{-1}d^{-1}ef^{-1}fe^{-1}dcba^{-1}$.
4. $abc^{-1}db^{-1}ea^{-1}fgd^{-1}g^{-1}hci^{-1}f^{-1}e^{-1}ih^{-1}$.
5. $a_1a_2\cdots a_na_1^{-1}a_2^{-1}\cdots a_n^{-1}$.
6. $a_1a_2\cdots a_na_1a_2\cdots a_n$.

Exercise 8.7.7. Use Exercise 8.7.4 and Theorem 8.7.4 to prove that the construction of the connected sum of connected closed surfaces is independent of the way the boundary circles are identified.

Exercise 8.7.8. What are the three spaces obtained by gluing two of squares in Figure 8.7.8 together.

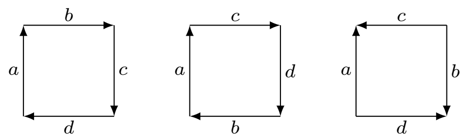



**Figure 8.7.8.** *square diagrams*

Exercise 8.7.9. What do you get from a "Möbius diagram" in Figure 8.7.9, in which the boundary circle of a Möbius band is identified as $aba^{-1}b^{-1}$? In general, for a surface with one circle as a boundary, what do you get by identifying the boundary circle in the same way as the boundary of a planar diagram?

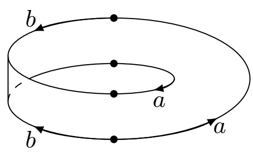



**Figure 8.7.9.** *Möbius diagram*

Exercise 8.7.10. Figure 8.7.10 is a surface with circle boundary, denoted $M_2$. The surface $M_k$ for $k \geq 1$ (also a surface with boundary) is the obvious generalization of $M_2$. What is the surface obtained by gluing $M_k$ and $M_l$ together along their boundary circles?

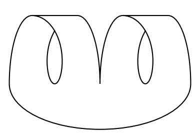


**Figure 8.7.10.** *a B-shaped surface with circle boundary*

Exercise 8.7.11. For any connected surfaces $S_1$ and $S_2$, remove two disjoint disks $B_1$ and $B_2$ from each surface. Glue the the boundary circle $\partial B_1$ in $S_1$ and the corresponding boundary circle $\partial B_1$ in $S_2$. Then do the same with the boundary circles $\partial B_2$ in $S_1$ and $S_2$. The result is a "double connected sum"

$$S_1\#\#S_2 = (S_1 - \mathring{B}_1 - \mathring{B}_2) \cup_{\partial B_1 \cup \partial B_2} (S_2 - \mathring{B}_1 - \mathring{B}_2).$$

What is $T^2\#\#T^2$? What is $T^2\#\#P^2$? In general, express $S_1\#\#S_2$ in terms of the usual connected sum.

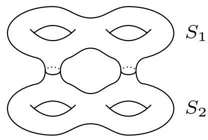



**Figure 8.7.11.** *double connected sum* $S_1 \# \# S_2$

Exercise 8.7.12. Figure 8.7.12 shows the "triple connected sum" of three surfaces, in which the boundary circles of $S_1 - \mathring{B}^2$, $S_2 - \mathring{B}^2$, $S_3 - \mathring{B}^2$ are glued together to form a graph of two vertices and three edges. Prove that the triple connected sum is well defined for connected surfaces. Moreover, find the triple connected sum of $K^2$, $T^2$ and $2T^2$?

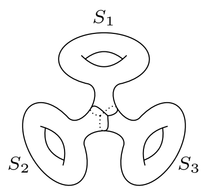



**Figure 8.7.12.** *triple connected sum*

## Chapter 9

# Topics in Point Set Topology

## 9.1 Normal Space

**Definition 9.1.1.** A topological space is *normal* if for any disjoint closed subsets $A$ and $B$, there are disjoint open subsets $U$ and $V$, such that $A \subset U$ and $B \subset V$.

In a normal space, disjoint closed subsets can be separated by disjoint open subsets. Like the Hausdorff property, the normal property is one of the separation axioms and is a topological property.

**Example 9.1.1.** All four topologies on the two point space are normal. In particular, normal spaces are not necessarily Hausdorff.

**Example 9.1.2.** Metric spaces are normal. Let $A$ and $B$ be disjoint closed subsets in a metric space. For any $a \in A$ and $b \in B$, we have $a \notin B$ and $b \notin A$. Since $A$ and $B$ are closed, there are $\epsilon_a > 0$ and $\epsilon_b > 0$, such that

$$B(a, \epsilon_a) \cap B = \emptyset, \quad B(b, \epsilon_b) \cap A = \emptyset.$$

Then $U = \cup_{a \in A} B\left(a, \frac{\epsilon_a}{2}\right)$ and $V = \cup_{b \in B} B\left(b, \frac{\epsilon_b}{2}\right)$ are open subsets containing $A$ and $B$ respectively. Moreover, if $U$ and $V$ have nonempty intersection, then for some $a \in A$ and $b \in B$, the balls $B\left(a, \frac{\epsilon_a}{2}\right)$ and $B\left(b, \frac{\epsilon_b}{2}\right)$ have a common point $c$. This implies

$$d(a,b) \le d(a,c) + d(c,b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \le \max\{\epsilon_a, \epsilon_b\}.$$

Therefore we have $a \in B(b, \epsilon_b)$ in case $\epsilon_a \le \epsilon_b$ or $b \in B(a, \epsilon_a)$ in case $\epsilon_b \le \epsilon_a$, contradicting to our construction.

**Example 9.1.3.** By Exercise 7.5.21, compact Hausdorff spaces are normal.

Exercise 9.1.1. Suppose a topology is normal. Is a finer topology also normal? What about a coarser topology?

Exercise 9.1.2. Prove that the lower limit topology is normal.

Exercise 9.1.3. Prove that the following are equivalent conditions for a space to be regular.

1. If a closed subset $C$ is contained in an open subset $U$, then there is an open subset $U'$ and a closed subset $C'$, such that $C \subset U' \subset C' \subset U$.
2. If a closed subset $C$ is contained in an open subset $U$, then there is an open subset $V$, such that $C \subset V$ and $\bar{V} \subset U$.
3. If $A$ and $B$ are disjoint closed subsets, then there are open subsets $U$ and $V$, such that $A \subset U$, $B \subset V$, and $\bar{U}$ and $\bar{V}$ are disjoint.

The first interpretation will be used in the proof of Theorem 9.1.2.

Exercise 9.1.4. Prove that if $A_1, A_2, \dots, A_n$ are mutually disjoint closed subsets in a normal space, then there are mutually disjoint open subsets $U_1, U_2, \dots, U_n$ such that $A_i \in U_i$. In fact, we can further make sure that the closures of $U_1, U_2, \dots, U_n$ are disjoint.

Exercise 9.1.5. Prove that (the subspace topology on) a closed subset of a normal space is normal.

Exercise 9.1.6. Prove that if $X, Y \neq \emptyset$ and $X \times Y$ is normal, then $X$ and $Y$ are normal.

Exercise 9.1.7. Consider $L = \{(x,y)\colon x+y=0\} \subset \mathbb{R}^2_{[\ )} = \mathbb{R}_{[\ )} \times \mathbb{R}_{[\ )}$.

1. Prove that any subset of $L$ is a closed subset.
2. Prove that $\overline{U \cap \mathbb{Q}^2} = \bar{U}$ for any open subset $U \subset \mathbb{R}^2_{[\ )}$.
3. Prove that if $\mathbb{R}^2_{[\ )}$ were normal, then for any subset $A \subset L$, there is an open subset $U_A$ satisfying $A \subset U_A$ and $\bar{U}_A \cap L = A$. Moreover, the map

$$\phi(A) = U_A \cap \mathbb{Q}^2 \colon \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$$

is injective.

Since $\mathcal{P}(L)$ contains more elements than $\mathcal{P}(\mathbb{Q}^2)$, there cannot be an injective map from $\mathcal{P}(L)$ to $\mathcal{P}(\mathbb{Q}^2)$. Therefore by Exercise 9.1.2, the product of normal spaces is not necessarily normal.

Exercise 9.1.8. Suppose $\{U_n \colon n \in \mathbb{N}\}$ and $\{V_n \colon n \in \mathbb{N}\}$ are countable open covers of disjoint closed subsets $A$ and $B$, such that $\bar{U}_n \cap B = \emptyset$ and $\bar{V}_n \cap A = \emptyset$ for any $n$. Denote

$$U_n' = U_n - \bar{V}_1 - \cdots - \bar{V}_n, \quad V_n' = V_n - \bar{U}_1 - \cdots - \bar{U}_n.$$

Prove that $\{U_n' \colon n \in \mathbb{N}\}$ and $\{V_n' \colon n \in \mathbb{N}\}$ are open covers of $A$ and $B$, such that $U_m' \cap V_n' = \emptyset$ for any $m$ and $n$. Then use this to further prove that second countable (see Example 5.2.7) regular spaces are normal.

The reason for introducing the concept of normal spaces is the following fundamental result.

**Theorem 9.1.2** (Urysohn Lemma). *Let $A$ and $B$ be disjoint closed subsets in a normal space $X$. Then there is a continuous function $f\colon X \to [0,1]$, such that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$.*

*Proof.* The function $f$ will be constructed by considering the "level subsets"

$$U_t = f^{-1}[0,t), \quad C_t = f^{-1}[0,t], \quad 0 \le t \le 1.$$

A function induces subsets $U_t$ and $C_t$ satisfying

$$U_t \subset C_t, \quad U_t = \cup_{r<t} U_r, \quad C_t = \cap_{r>t} C_r, \quad C_t \subset U_{t'} \text{ for } t < t', \quad U_0 = \emptyset, \quad C_1 = X.$$

Conversely, if we have subsets $U_t$ and $C_t$ satisfying the conditions above, then a set-theoretical argument gives a partition $X = \sqcup_{0<t\le 1}(C_t - U_t)$, and

$$f(x) = t \text{ if } x \in C_t - U_t$$

defines a function with $U_t$ and $C_t$ as the level sets.

The continuity of $f$ means that $U_t$ is open and $C_t$ is closed. Moreover, the properties $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$ mean that

$$A \subset C_0, \quad B \cap U_1 = \emptyset.$$

Therefore the construction of $f$ is the same as constructing open subsets $U_t$ and closed subsets $C_t$ satisfying all the conditions above.

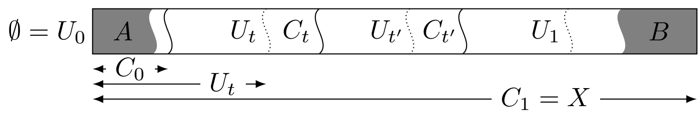



**Figure 9.1.1.** *levels of a continuous map*

Our strategy is to construct open subsets $V_r$ and closed subsets $D_r$ for $r$ in a dense subset $R$ of $[0,1]$, such that $0, 1 \in R$ and the similar conditions for $U_t$ and $C_r$ are satisfied

$$V_r \subset D_r, \quad D_r \subset V_{r'} \text{ for } r < r', \quad V_0 = \emptyset, \quad D_0 = A, \quad V_1 = X - B, \quad D_1 = X.$$

Then for any $0 \le t \le 1$, the subset $U_t = \cup_{r<t, r\in R} V_r$ is open, the subset $C_t = \cap_{r>t, r\in R} D_r$ is closed. Moreover, a set-theoretical argument shows that the subsets satisfy all the conditions for constructing the desired continuous function.

Take

$$R = \cup_{n=0}^{\infty} R_n, \quad R_n = \left\{ \frac{k}{2^n} : 0 \le k \le 2^n \right\}.$$

We have $R_n \subset R_{n+1}$ and $R$ is dense in $[0,1]$. For $R_0 = \{0,1\}$, take open subsets $V_0 = \emptyset$, $V_1 = X - B$ and closed subsets $D_0 = A$, $D_1 = X$. They satisfy

$$V_r \subset D_r, \quad D_r \subset V_{r'} \quad \text{for } r < r' \text{ and } r, r' \in R_0,$$

Suppose open subsets $V_r$ and closed subsets $C_r$ have been constructed for $r \in R_n$, such that the following is satisfied

$$V_r \subset D_r, \quad D_r \subset V_{r'} \quad \text{for } r < r' \text{ and } r, r' \in R_n.$$

For any $r \in R_{n+1} - R_n$, we have

$$r_- = \frac{k}{2^n} < r = \frac{2k+1}{2^{n+1}} < r_+ = \frac{k+1}{2^n}, \text{ with } r_-, r_+ \in R_n.$$

By the inductive assumption on the construction for $R_n$, we have already constructed an open subset $V_{r_+}$ and a closed subset $D_{r_-}$ satisfying $D_{r_-} \subset V_{r_+}$. By the first interpretation of the normal property in Exericse 9.1.3, we can find an open subset $V_r$ and a closed subset $D_r$, such that

$$D_{r_-} \subset V_r \subset D_r \subset V_{r_+}.$$

This implies that the following is satisfied

$$V_r \subset D_r, \quad D_r \subset V_{r'} \quad \text{for } r < r' \text{ and } r, r' \in R_{n+1}.$$

At the end of the construction, we see the two conditions are satisfied for all $r < r'$ and $r, r' \in R$. $\square$

Theorem 9.1.2 has the following useful extension. The proof for the bounded case is outlined in Exercise 9.1.14. The proof for the general case is given in Exercise 9.1.15.

**Theorem 9.1.3** (Tietze[27] Extension Theorem). *Any continuous function on a closed subset in a normal space $X$ can be extended to a continuous function on the whole space.*

Exercise 9.1.9. For metric spaces, use the function $d(x, A)$ in Exercise 2.4.15 to give a direct construction of the function $f$ in Urysohn lemma.

Exercise 9.1.10. Prove the converse of Urysohn lemma: Suppose for any disjoint open subsets $A$ and $B$, there is a continuous function $f\colon X \to [0, 1]$, such that $f = 0$ on $A$ and $f = 1$ on $B$. Then $X$ is normal.

Exercise 9.1.11. Prove that a connected normal Hausdorff space with more than one point must be uncountable.

Exercise 9.1.12. Let $A_1$, $A_2$, ..., $A_n$ be mutually disjoint closed subsets in a normal space $X$. Let $a_1$, $a_2$, ..., $a_n$ be any numbers. Prove that there is a continuous function $f\colon X \to \mathbb{R}$, such that $f = a_i$ on $A_i$.

Exercise 9.1.13. Let $A$ be a closed subset of a normal space $X$. Prove that $A = f^{-1}(1)$ for some continuous function on $X$ if and only if $A$ is the intersection of countably many open subsets.

Exercise 9.1.14. Let $A$ be a closed subset of a normal space $X$. Let $f\colon A \to \mathbb{R}$ be a continuous function satisfying $|f(x)| \le b$ for any $x \in A$. Prove Tietze extension theorem in the following steps.

1. By considering disjoint closed subsets $f^{-1}\left[\dfrac{b}{3}, b\right]$ and $f^{-1}\left[-b, -\dfrac{b}{3}\right]$, construct a continuous function $g\colon X \to \mathbb{R}$ satisfying
$$|f - g| \le \frac{2b}{3} \text{ on } A, \quad |g| \le \frac{b}{3} \text{ on } X.$$
2. Construct a sequence of continuous functions $g_n\colon X \to \mathbb{R}$ satisfying
$$|f - g_1 - \cdots - g_n| \le \frac{2^n b}{3^n} \text{ on } A, \quad |g_n| \le \frac{2^{n-1} b}{3^n} \text{ on } X.$$

[27] Heinrich Franz Friedrich Tietze, born August 31, 1880 in Schleinz, Austria, died February 17, 1964 in Munich, Germany. Tietze made numerous contributions to topology and introduced Tietze transformation between different presentations of a group.

3. Prove that if $g_n : X \to \mathbb{R}$ are continuous functions satisfying $|g_n| < b_n$ for constants $b_n$, and $\sum b_n$ converges, then the series $\sum g_n$ converges to a continuous function.
4. Prove that for the sequence $g_n$ constructed in the second part, the series $\sum g_n$ converges and defines a continuous function $g$ on $X$. Moreover, prove that $g = f$ on $A$ and $|g| \le b$ on $X$.

Exercise 9.1.15. Let $A$ be a closed subset of a normal space $X$. Let $f : A \to (0, 1)$ be a continuous function.

1. Prove that there is a continuous function $g : X \to [0, 1]$ extending $f$.
2. Prove that there is a continuous function $h : X \to [0, 1]$ such that $h = 1$ on $A$ and $g(x) = 0$ implies $h(x) = 0$.
3. Prove that the product $gh$ is a continuous function extending $f$ and has values in $(0, 1)$.
4. Prove Tietze extension theorem for the general case that $f$ may not be bounded.

Exercise 9.1.16. Use Tietze extension theorem to prove Urysohn lemma.

Exercise 9.1.17. Prove the following are equivalent for a metrizable space $X$.

1. $X$ is compact.
2. Every metric on $X$ is bounded.
3. Every continuous function on $X$ is bounded.

## 9.2 Paracompact Space

A collection $\mathcal{A}$ of subsets is a *refinement* of another collection $\mathcal{B}$ if any subset in $\mathcal{A}$ is contained in some subset in $\mathcal{B}$.

A collection $\mathcal{A}$ of subsets in a topological space $X$ is *locally finite* if any point has a neighborhood $N$, such $A \cap N = \emptyset$ for all but finitely many $A \in \mathcal{A}$.

**Definition 9.2.1.** A topological space is *paracompact* if every open cover has a locally finite open cover refinement.

Compact spaces are clearly paracompact. A highly non-trivial theorem by A. H. Stone says that metric spaces are paracompact.

Exercise 9.2.1. Find an open cover of $\mathbb{R}$ in the usual topology, such that every point belongs to finitely many subsets in the cover, but the cover is not locally finite. This shows that pointwise finiteness does not imply local finiteness.

Exercise 9.2.2. Find an open cover of $\mathbb{R}$ in the usual topology with no locally finite subcover. This shows that refinement cannot be replaced by subcover in the definition of paracompactness.

Exercise 9.2.3. Prove that a locally finite cover of a compact space is a finite cover. Moreover, a locally finite cover of a second countable space is a countable cover.

Exercise 9.2.4. Let $\mathcal{U}$ be a locally finite open cover of $X$.

1. Suppose for each $U \in \mathcal{U}$, we have a locally finite cover $\mathcal{V}_U$ of $U$. Prove that $\cup_{U\in\mathcal{U}}\mathcal{V}_U$ is a locally finite cover of $X$.
2. Prove that if the open subsets in $\mathcal{U}$ are paracompact, then $X$ is also paracompact.

In particular, this shows that the Euclidean spaces are paracompact.

Exercise 9.2.5. Prove that if a collection $\{A_i\}$ is locally finite, then $(\cup A_i)' = \cup A_i'$, $\overline{\cup A_i} = \cup\bar{A}_i$, and the collection $\{\bar{A}_i\}$ is also locally finite. In particular, the union of a locally finite collection of closed subsets is closed.

Exercise 9.2.6. Prove that a closed subset of a paracompact space is still paracompact.

Exercise 9.2.7. Prove that all subsets of a topological space are paracompact if and only if all open subsets are paracompact.

Exercise 9.2.8. A subset $A$ of a topological space $X$ may be covered by open subsets in $X$ or by open subsets in $A$. Note that open subsets in $A$ are used in the definition of paracompactness of $A$.

1. Prove that if any cover of $A$ by open subsets in $X$ has a locally finite refinement cover by open subsets in $X$, then $A$ is paracompact. In fact, the local finiteness only needs to be verified around points in $A$.
2. Prove that if $A$ is open and the subspace topology is paracompact, then any cover of $A$ by open subsets in $X$ has a locally finite refinement cover by open subsets in $X$.

The result should be compared with compactness, when the two types of open covers give equivalent definition.

Exercise 9.2.9. Suppose there is a sequence of open subsets $U_n$ of $X$, such that $\bar{U}_n \subset U_{n+1}$ and $X = \cup U_n$. Prove that if $\bar{U}_n$ are paracompact, then $X$ is paracompact.

Exercise 9.2.10. Suppose there is a locally finite open cover $\mathcal{U}$ of $X$. Prove that if every $U \in \mathcal{U}$ is contained in a paracompact subspace, then $X$ is paracompact.

Exercise 9.2.11. We try to prove that the lower limit topology is paracompact. First explain that it is sufficient to prove that $[0,1)$ is paracompact in the lower limit topology. Then for any cover $\mathcal{U}$ of $[0,1)$ by open subsets in the lower limit topology, denote

$$A = \{x \colon [0,x) \text{ is covered by a locally finite open refinement of } \mathcal{U}\}.$$

1. Prove that $A$ is not empty.
2. Prove that for any $x \in A$ and $0 < y < x$, $[y,x)$ is covered by a locally finite refinement of $\mathcal{U}$ consisting of open subsets in $[y,x)$.
3. Prove that $\sup A \in A$ and $\sup A = 1$.

From the proof of the second property in Theorem 7.5.2 and Exercise 7.5.21, we know compact Hausdorff spaces are regular and normal. By essentially the same proof, the fact can be extended to paracompact Hausdorff spaces.

**Proposition 9.2.2.** *Paracompact Hausdorff spaces are regular and normal.*

*Proof.* Suppose $A$ is a closed subset of a paracompact Hausdorff space $X$ and $x \notin A$. Since $X$ is Hausdorff, for each $a \in A$, there are disjoint open subsets $U_a$ and $V_a$, such that $x \in U_a$ and $a \in V_a$. Since $X$ is paracompact, the open cover $\{V_a : a \in A\} \cup \{X - A\}$ of $X$ has a locally finite refinement open cover. Taking those in the refinement that have non-empty intersections with $A$, we get a locally finite refinement open cover $\mathcal{V}$ of $A$. Then $W = \cup_{V \in \mathcal{V}} V$ is an open subset containing $A$.

We claim that $x \notin \bar{W}$, so that $X - \bar{W}$ and $W$ are disjoint open subsets separating $x$ and $A$. Since $\mathcal{V}$ is locally finite, we have $\bar{W} = \cup_{V \in \mathcal{V}} \bar{V}$ (see Exercise 9.2.5). Moreover, any $V \subset V_a$ for some $a \in A$. Therefore $x \in U_a$ and $U_a$ is disjoint from $V$. Since $U_a$ is open, we get $x \notin \bar{V}$. Therefore $x \notin \bar{W}$.

Once the regular property is established, the normal property can be similarly proved, like Exercise 7.5.21. □

Now we come to the main reason of introducing the paracompact property. For any function $\phi : X \to \mathbb{R}$, the *support* of the function is

$$\mathrm{supp}(\phi) = \overline{\{x : \phi(x) \neq 0\}}.$$

**Definition 9.2.3.** Let $\mathcal{U}$ be an open cover of $X$. A *partition of unity* dominated by $\mathcal{U}$ is a collection of continuous functions $\phi_U : X \to [0, 1]$ indexed by $U \in \mathcal{U}$, such that

1. $\mathrm{supp}(\phi_U) \subset U$ for any $U \in \mathcal{U}$.
2. The collection $\{\mathrm{supp}(\phi_U)\}$ of supports is locally finite.
3. $\sum_{U \in \mathcal{U}} \phi_U = 1$.

The first condition implies $\phi_U = 0$ outside $U$. The second condition implies that for any point $x$, the sum $\sum_{U \in \mathcal{U}} \phi_U(x)$ in the third condition makes sense because only finitely many terms are non-zero.

**Theorem 9.2.4.** *Any open cover in a paracompact Hausdorff space admits a partition of unity.*

The proof of the theorem is based on the fact that, with the additional Hausdorff condition, the refinement open cover in paracompact spaces can be more sophisticated.

**Proposition 9.2.5.** *Any open cover $\mathcal{U}$ of a paracompact Hausdorff space has a locally finite refinement $\mathcal{V}$, such that the closure of any open subset in $\mathcal{V}$ is contained in some open subset in $\mathcal{U}$.*

*Proof.* Suppose $\mathcal{U}$ is an open cover of a paracompact Hausdorff space. Consider

$$\mathcal{W} = \{W : W \subset X \text{ is open and } \bar{W} \subset U \text{ for some } U \in \mathcal{U}\}.$$

We claim that $\mathcal{W}$ is an open cover. Any $x \in X$ is contained in some $U \in \mathcal{U}$. Then $x \notin X - U$ and $X - U$ is closed. By Proposition 9.2.2, $x$ and $X - U$ may be separated by disjoint open subsets. Therefore there is an open subset $W$, such that $x \in W$ and $\bar{W} \cap (X - U) = \emptyset$. We get $x \in W \in \mathcal{W}$.

Since $X$ is paracompact, the open cover $\mathcal{W}$ has a locally finite subcover $\mathcal{V}$. By the way $\mathcal{W}$ is defined, the closure of any subset in $\mathcal{V}$ is contained in some subset in $\mathcal{U}$. □

The conclusion of the proposition needs to be further enhanced a little bit in order to prove the main theorem. Suppose a locally finite open cover $\mathcal{V} = \{V_j : j \in J\}$ refines an open cover $\mathcal{U} = \{U_i : i \in I\}$ in the way described in the proposition. Then we have a map $\alpha: J \to I$, such that $\bar{V}_j \subset U_{\alpha(j)}$. For any $i \in I$, let $V^i = \cup_{\alpha(j)=i} V_j$. Since $\mathcal{V}$ is locally finite, by Exercise 9.2.5, we have

$$\bar{V}^i = \cup_{\alpha(j)=i} \bar{V}_j \subset U_i.$$

Therefore $\{V^i : i \in I\}$ is also a locally finite open cover satisfying $\bar{V}^i \subset U_i$. The enhancement is that the refinement is indexed in the same way as $\mathcal{U}$. Then the conclusion of the proposition may be interpreted as the following: Any open cover can be "properly shrunken" to become a locally finite open cover, where by properly shrinking $U$ to $V$ we mean $\bar{V} \subset U$. We also understand that some $U$ may be shrunken to the empty set, which happens when $i$ is not in the image of the index map $\alpha$.

*Proof of Theorem 9.2.4.* Let $\mathcal{U}$ be an open cover of a paracompact Hausdorff space $X$. Then $\mathcal{U}$ can be properly shrunken to a locally finite open cover $\mathcal{V}$. The open cover $\mathcal{V}$ can be further properly shrunken to a locally finite open cover $\mathcal{W}$. In other words, for each $U \in \mathcal{U}$, we have open subsets $V$ and $W$, such that

$$\bar{W} \subset V \subset \bar{V} \subset U.$$

Moreover, all such $V$ form $\mathcal{V}$, and all such $W$ form $\mathcal{W}$.

By Proposition 9.2.2, $X$ is normal. For each $U$, the corresponding $\bar{W}$ and $X - V$ are disjoint closed subsets. By Theorem 9.1.2, there is a continuous function $\psi_U : X \to [0, 1]$, such that $\psi_U = 1$ on $\bar{W}$ and $\psi_U = 0$ on $X - V$. We have $\text{supp}(\psi_U) \subset \bar{V} \subset U$. By Exercise 9.2.5, the collection $\{\bar{V} : V \in \mathcal{V}\}$ is still locally finite. Therefore the collection $\{\text{supp}(\psi_U)\}$ is also locally finite. Moreover, since $\mathcal{W}$ is a cover, we have $\sum \psi_U \geq 1$ everywhere. Then

$$\phi_U = \frac{\psi_U}{\sum \psi_U}$$

is a partition of unity dominated by $\mathcal{U}$. □

Exercise 9.2.12. By Exercise 9.2.11, the lower limit topology is paracompact. However, explain that the product of the lower limit topology with itself is not paracompact.

Exercise 9.2.13. Prove that the product of a paracompact space and a compact space is still paracompact.

Exercise 9.2.14. Prove that paracompact Hausdorff spaces also have the "expansion property": If $\mathcal{A}$ is a locally finite collection of subsets, then for each $A \in \mathcal{A}$, there is an open subset $U$ containing $A$, such that all such $U$ form a locally finite collection.

## 9.3 Complete Metric Space

In a metric space, a sequence $a_n$ converges to $x$ if for any $\epsilon > 0$, there is $N$, such that

$$n > N \implies d(a_n, x) < \epsilon.$$

It is easy to show that a convergent sequence satisfies the *Cauchy criterion*: For any $\epsilon > 0$, there is $N$, such that

$$m, n > N \implies d(a_m, a_n) < \epsilon.$$

A sequence satisfying the Cauchy criterion is called a *Cauchy sequence*. In real analysis, we know a sequence of real numbers converges if and only if it is a Cauchy sequence. The fact does not hold if the metric space is the set of rational numbers, because the limit may be irrational and does not lie in the space itself. Therefore the following property can be used to distinguish real and rational numbers.

**Definition 9.3.1.** A metric space is *complete* if any Cauchy sequence converges (to a point within the space).

**Example 9.3.1.** Discrete metric spaces are complete. Finite metric spaces are complete. In both cases, we have a number $\epsilon > 0$ such that $d(x, y) \geq \epsilon$ for any distinct $x, y$. If we choose this $\epsilon$ in the Cauchy criterion, we get $a_m = a_n$ for $m, n > N$. Therefore the sequence is essentially constant and converges.

**Example 9.3.2.** The uniform convergence of a sequence of functions is measured by the $L_\infty$-metric. It is a classical result in mathematical analysis that Cauchy criterion can be applied to the uniform convergence, and the limit of a uniformly convergent sequence of continuous functions is still continuous. Therefore $C[0, 1]$ is complete under the $L_\infty$-metric.

Under the $L_1$-metric, it is possible to find a sequence of continuous functions that converges to a non-continuous function. One such example is $f_n(t) = t^n$ for $t \in [0, 1]$. The situation is similar to a sequence of rational (comparable to continuous) numbers converging to an irrational (comparable to non-continuous) number. Therefore $C[0, 1]$ is not complete under the $L_1$-metric.

Exercise 9.3.1. Prove that if $a_n$ is a Cauchy sequence and $\lim_{n\to\infty} d(a_n, b_n) = 0$, then $b_n$ is also a Cauchy sequence.

Exercise 9.3.2. Prove that a subset of a complete metric space is complete if and only if the subset is closed.

Exercise 9.3.3. Prove that the product of two non-empty metric spaces is complete if and only if each metric space is complete.

Exercise 9.3.4. Suppose two metrics satisfy $d_1(x,y) \le c\,d_2(x,y)$. Is there any relation between the completeness of $d_1$ and $d_2$?

Exercise 9.3.5. Prove that a Cauchy sequence converges if and only if it has a convergent subsequence. In particular, a metric space is complete if and only if every Cauchy sequence has a convergent subsequence.

Exercise 9.3.6 (Nesting Principle). Prove that a metric space $X$ is complete if any only if the following holds: If $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ is a decreasing sequence of closed subsets, such that the diameter (see Exercise 2.2.7) of $C_n$ converges to 0, then $\cap C_n \neq \emptyset$.

Exercise 9.3.7. Prove that compact metric spaces are complete. However, locally compact (see Exercise 7.5.29) metric spaces may not be complete.

Exercise 9.3.8. Prove that if there is $\epsilon > 0$, such that the closure of any ball of radius $\epsilon$ is compact, then the metric space is complete.

Exercise 9.3.9. Let $U$ be an open subset of a complete metric space $X$. Then

$$\phi(x) = \frac{1}{d(x, X-U)} \colon U \to (0, +\infty)$$

is continuous. Let $\tilde{d}(x,y) = d(x,y) + |\phi(x) - \phi(y)|$ for $x, y \in U$.

1. Prove that $\tilde{d}$ is a complete metric on $U$.
2. Prove that $\tilde{d}$ induces the subspace topology on $U$.

We see that the subspace topology on any open subset of a complete metric space is induced by a complete metric, although the metric may not be the original one.

Exercise 9.3.10 (Contraction Principle). Let $X$ be a complete metric space. Let $f\colon X \to X$ be a map satisfying $d(f(x), f(y)) \le c\,d(x,y)$ for some constant $c < 1$.

1. Prove that for any $x$, the sequence $f(x), f(f(x)), f(f(f(x))), \ldots$, converges.
2. Prove that the limit $a$ of the sequence is a fixed point: $f(a) = a$.
3. Prove that the fixed point of $f$ is unique.

Exercise 9.3.11 (Baire[28] Metric). Let $S$ be any set and let $S^{\mathbb{N}}$ be the set of sequences in $S$. For any $x = \{X_n\}, y = \{Y_n\} \in S^{\mathbb{N}}$, define the *Baire metric* $d(x,y) = \dfrac{1}{n}$, where $n$ is the smallest number such that $X_n \neq Y_n$. Prove that $d$ is a complete metric.

Exercise 9.3.12. Any irrational number in $[0,1]$ can be expressed as an infinite continued fraction

$$x = \cfrac{1}{X_1 + \cfrac{1}{X_2 + \cfrac{1}{X_3 + \ddots}}}, \quad X_n \in \mathbb{N}.$$

[28] René-Louis Baire, born January 21, 1874 in Paris, France, died July 5, 1932 in Chambéry, France. Baire's work were mainly on the limit of functions, the continuity and the irrational numbers.

This gives a one-to-one correspondence between the set $\mathbb{R} - \mathbb{Q}$ of irrational numbers and the set $\mathbb{N}^{\mathbb{N}}$ in Exercise 9.3.11. Prove that under the correspondence, the usual topology on $\mathbb{R} - \mathbb{Q}$ is homeomorphic to the topology induced by the Baire metric on $\mathbb{N}^{\mathbb{N}}$. In particular, the usual topology on the irrational numbers is induced by a complete metric.

Exercise 9.3.13. Let $(Y, d)$ be a complete metric space, such that the value of $d$ is bounded. For any set $X$, the set $\mathcal{F}(X, Y)$ is the collection of all maps from $X$ to $Y$. Prove that the $L_\infty$-metric

$$d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$$

is a complete metric on $\mathcal{F}(X, Y)$.

Exercise 9.3.14. Let $(Y, d)$ be a complete metric space, such that the value of $d$ is bounded. Let $X$ be a topological space, and let the set $\mathcal{C}(X, Y)$ be the collection of all continuous maps from $X$ to $Y$. Prove that the $L_\infty$-metric is also complete on $\mathcal{C}(X, Y)$.

A given metric space $X$ may not be complete because some Cauchy sequences may not have limit in the space. By externally adding the limit points of Cauchy sequences, the space $X$ can be enlarged to a space $\bar{X}$ such that all Cauchy sequences converge and all points in $\bar{X}$ are the limit points of sequences in $X$.

**Definition 9.3.2.** Given a metric space $X$, a *completion* of $X$ is a complete metric space $\bar{X}$ that contains $X$ as a dense subset and extends the metric of $X$.

For example, with the usual metric, $\mathbb{R}$ is the completion of $\mathbb{Q}$, $[0, 1]$ is the completion of $(0, 1)$. Moreover, with the $L_\infty$-metric, $C[0, 1]$ is the completion of the space of all polynomials. On the other hand, with the $L_1$-metric, the completion of $C[0, 1]$ is the space $L_1[0, 1]$ of Lebesgue integrable functions on $[0, 1]$, subject to the equivalence relation that $f \sim g$ if and only if $f = g$ almost everywhere.

**Proposition 9.3.3.** *Any metric space $X$ has a unique completion.*

*Proof.* For any two Cauchy sequences $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ in $X$, $d(a_n, b_n)$ is a Cauchy sequence of numbers. Define the distance between Cauchy sequences by

$$\bar{d}(\alpha, \beta) = \lim_{n \to \infty} d(a_n, b_n).$$

It is easy to see that $\bar{d}$ is non-negative and satisfies the symmetry and the triangle inequalities. As a result, the condition $\bar{d}(\alpha, \beta) = 0$ defines an equivalence relation among all Cauchy sequences. Let $\bar{X}$ be the set of equivalence classes of Cauchy sequences. Then $\bar{d}$ is a well-defined metric on $\bar{X}$. See Exercise 2.1.9 for more details.

For any $a \in X$, denote by $i(a)$ the constant sequence in which each term is $a$. Then $i$ is a map from $X$ to $\bar{X}$ satisfying $d(a, b) = \bar{d}(i(a), i(b))$. Therefore $i$ isometrically embeds $X$ into $\bar{X}$.

A point in $\bar{X}$ is given by a Cauchy sequence $\alpha = \{a_n\}$. For any $\epsilon > 0$, there is $N$, such that $m, n \geq N$ implies $d(a_m, a_n) < \epsilon$. This implies that for any $n \geq N$, we have $\bar{d}(\alpha, i(a_n)) = \lim_{m\to\infty} d(a_m, a_n) \le \epsilon$. This proves that

$$\lim_{n\to\infty} i(a_n) = \alpha.$$

In particular, we conclude that $X$ is dense in $\bar{X}$.

Next we prove that $\bar{X}$ is complete. Consider a Cauchy sequence $\alpha^1 = \{a_n^1\}, \alpha^2 = \{a_n^2\}, \dots,$ in $\bar{X}$. For each fixed $k$, we use $\lim_{n\to\infty} i(a_n^k) = \alpha^k$ to find a term $a_{n_k}^k$ in the Cauchy sequence that defines $\alpha^k$, such that $i(a_{n_k}^k)$ is very close to $\alpha^k$. Then we get a sequence $i(a_{n_1}^1), i(a_{n_2}^2), \dots,$ satisfying

$$\lim_{k\to\infty} \bar{d}(i(a_{n_k}^k), \alpha^k) = 0.$$

By Exercise 9.3.1 and the assumption that $\alpha^1, \alpha^2, \dots,$ is a Cauchy sequence, we see that $i(a_{n_1}^1), i(a_{n_2}^2), \dots,$ is a Cauchy sequence. Since $i$ is an isometry, the sequence $a_{n_1}^1, a_{n_2}^2, \dots,$ in $X$ is also Cauchy and represents a point $\alpha$ in $\bar{X}$. We claim that $\lim_{m\to\infty} \alpha^m = \alpha$. The proof is based on the estimation

$$\bar{d}(\alpha, \alpha^m) \le \bar{d}(\alpha, i(a_{n_k}^k)) + \bar{d}(i(a_{n_k}^k), \alpha^k) + \bar{d}(\alpha^k, \alpha^m).$$

For any $\epsilon > 0$, there is $N$, such that $k, m > N$ implies $\bar{d}(\alpha^k, \alpha^m) < \epsilon$. Now we fix $m > N$ and take $k \to \infty$ in the estimation above. Since $\alpha$ is defined by the sequence $a_{n_1}^1, a_{n_2}^2, \dots,$ we have $\lim_{k\to\infty} \bar{d}(\alpha, i(a_{n_k}^k)) = 0$. We also have $\lim_{k\to\infty} \bar{d}(i(a_{n_k}^k), \alpha^k) = 0$ by our construction. We conclude that $m > N$ implies $\bar{d}(\alpha, \alpha^m) \le \epsilon$. This proves the claim and the completeness of $\bar{X}$.

The proof of the uniqueness of the completion follows from the universal property satisfied by the completion. See Exercises 9.3.19 and 9.3.20. □

Exercise 9.3.15. Suppose two metrics satisfy $d_1(x, y) \le c\, d_2(x, y)$. Is there any relation between the completions with respect to $d_1$ and $d_2$?

Exercise 9.3.16. Prove that the completions with respect to $d$, $\min\{d, 1\}$, $\sqrt{d}$ are the same.

Exercise 9.3.17. What is the completion of the product of two metric spaces?

Exercise 9.3.18. Prove that if $X$ is a subset of a complete metric space $Y$, then the closure $\bar{X}$ of $X$ in $Y$ is the completion of $X$.

Exercise 9.3.19. A map $f\colon X \to Y$ between metric spaces is *uniformly continuous* if for any $\epsilon > 0$, there is $\delta > 0$, such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. Prove that if $Y$ is complete, then any uniformly continuous map $f\colon X \to Y$ has unique continuous extension $\bar{f}\colon \bar{X} \to Y$. Moreover, $\bar{f}$ is still uniformly continuous.

Exercise 9.3.20. Use Exercise 9.3.19 to prove the uniqueness of completion: If $Y$ and $Z$ are completions of $X$, then there is an invertible isometry $f\colon Y \to Z$ such that $f(x) = x$ for $x \in X$.

Exercise 9.3.21. For a metric space $X$ with the distance function $d$ bounded by $B$, prove that

$$i(x) = d(x, ?)\colon X \to \mathcal{C}(X, [0, B])$$

is an isometric embedding, where the space of continuous maps has the $L_\infty$-metric. By using Exercises 9.3.13 and 9.3.18, prove the existence of a completion of $X$. Furthermore, by using Exercise 9.3.16, extend the completion to the case when $d$ is not necessarily bounded.

Exercise 9.3.22 ($p$-adic Completion). Let $p$ be a prime. A *$p$-adic number* is a formal "Laurent series"

$$x = \sum_{n=k}^{\infty} a_n p^n = a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} + \cdots, \quad a_i \in \{0, 1, \dots, p-1\}.$$

The distance $d_{p\text{-adic}}(x, y)$ between two $p$-adic numbers $x$ and $y$ is $p^{-c}$, where $p^c$ is the smallest power for which the coefficients of $x$ and $y$ are different. Note that we may also allow $a_i$ to be non-negative integers and by expressing $a_i$ as polynomials in $p$, rearrange the expression to become the standard form with $0 \le a_i < p$.

We denote by $\mathbb{Q}_p$ all the $p$-adic numbers. Moreover, $x$ is a *$p$-adic integer* if $k \ge 0$, and we denote by $\mathbb{Z}_p$ all the $p$-adic integers.

1. Prove that $d_{p\text{-adic}}$ is a complete metric on $\mathbb{Q}_p$.
2. Introduce addition and multiplication among $p$-adic numbers and prove that the operations make $\mathbb{Q}_p$ into a field.
3. Prove that the algebraic operations are continuous with respect to the $p$-adic metric.
4. By expressing natural numbers as polynomials in $p$ (i.e., $p$-based expansion), construct a natural isometric embedding $\mathbb{N} \to \mathbb{Q}_p$.
5. Prove that the isometric embedding $\mathbb{N} \to \mathbb{Q}_p$ has unique extension to an isometric embedding $\mathbb{Q} \to \mathbb{Q}_p$ preserving the algebraic operations.
6. Use the isometric embedding to show that the $p$-adic completions of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are $\mathbb{Z}_p$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

## 9.4 Baire Category Theorem

In analysis, the size of a subset can be described by measure. For many purposes, a subset is negligible when the measure is zero, and is substantial when the measure is positive. For example, if the place where something does not happen form a subset of measure zero, then we say something happens almost everywhere. Moreover, if the places where something happens form a set of positive measure, then we are certain that something really happens somewhere.

The topology can also be used to describe negligible or substantial subsets. A subset $A$ can often be neglected if it is *nowhere dense*, which means that $A$ is not dense in any nonempty open subset $U$. In other words, the closure $\bar{A}$ does not contain any nonempty open subset. Recall the *interior* of $A$ is (see Exercise 4.6.11)

$$\mathring{A} = \cup\{U \subset A \colon U \text{ open}\} = X - \overline{X - A}.$$

Therefore $A$ is nowhere dense if and only if the closure $\bar{A}$ has empty interior. In other words, $X - \bar{A}$ is dense.

**Example 9.4.1.** Suppose a space is Fréchet and has no isolated points. In other words, any single point is closed but not open. Then any finite subset is closed and not open. Therefore any finite subset is nowhere dense. On the other hand, the countable subset of rational numbers is dense in the real number with the usual topology.

**Example 9.4.2 (Cantor[29] Set).** The Cantor set is obtained by successively removing open middle one third of intervals from the interval $[0,1]$. Specifically, we remove the open middle one third interval $\left(\frac{1}{3},\frac{2}{3}\right)$ and get two disjoint closed intervals of length $\frac{1}{3}$

$$K_1 = [0,1] - \left(\frac{1}{3},\frac{2}{3}\right) = \left[0,\frac{1}{3}\right] \sqcup \left[\frac{2}{3},1\right].$$

Then we remove the open middle one third of each of the two remaining intervals and get four disjoint closed intervals of length $\frac{1}{3^2}$

$$\begin{aligned} K_2 &= \left(\left[0,\frac{1}{3}\right] - \left(\frac{1}{3^2},\frac{2}{3^2}\right)\right) \sqcup \left(\left[\frac{2}{3},1\right] - \left(\frac{7}{3^2},\frac{8}{3^2}\right)\right) \\ &= \left[0,\frac{1}{3^2}\right] \sqcup \left[\frac{2}{3^2},\frac{3}{3^2}\right] \sqcup \left[\frac{6}{3^2},\frac{7}{3^2}\right] \sqcup \left[\frac{8}{3^2},1\right]. \end{aligned}$$

Keep going, we get $K_n$ after the $n$-th step, which is a disjoint union of $2^n$ closed intervals of length $\frac{1}{3^n}$. After deleting the open middle one third from each of these intervals, we get $K_{n+1}$. The *Cantor set* is the intersection of all

$$K = \cap_n K_n.$$

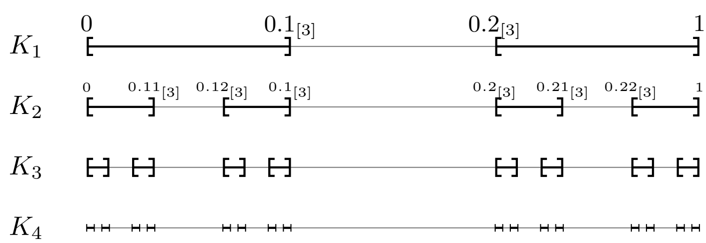



**Figure 9.4.1.** *construction of Cantor set*

In terms of the base 3 numerical system, the middle one third $\left(\frac{1}{3},\frac{2}{3}\right)$ of $[0,1]$ consists of numbers of the form (the subscript [3] indicates base 3)

$$\frac{1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots = 0.1a_2a_3\cdots_{[3]}, \quad a_i \in \{0,1,2\}, \quad \text{some } a_i \neq 2.$$

[29]Georg Ferdinand Ludwig Cantor, born March 3, 1845 in Saint Petersburg, Russia, died January 6, 1918 in Halle, Germany. Cantor's theory of infinite revolutionized the mathematics thinking. He introduced the famous Cantor set in 1883.

Similarly, the interval $\left(\frac{1}{3^2}, \frac{2}{3^2}\right)$ consists of numbers of the form $0.01a_3a_4\cdots_{[3]}$, and the interval $\left(\frac{7}{3^2}, \frac{8}{3^2}\right)$ consists of numbers of the form $0.21a_3a_4\cdots_{[3]}$. In general, it is not hard to see that the deleted intervals consist of numbers of the form $0.a_1a_2a_3\cdots_{[3]}$ in which some $a_i = 1$. Therefore as the complement, the Cantor set consists of $0.a_1a_2a_3\cdots_{[3]}$ in which each $a_i$ is either 0 or 2

$$K = \left\{\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots : a_i = 0 \text{ or } 2\right\}.$$

The Cantor set is nowhere dense. We only need to check that $K$ is not dense in the interval ($a_i$ are fixed and $z_j$ are arbitrary, the subscript [3] is omitted)

$$\begin{aligned} U &= (0.a_1a_2a_3\cdots a_n, 0.a_1a_2a_3\cdots a_n + 3^{-n}) \\ &= \{0.a_1a_2a_3\cdots a_nz_{n+1}z_{n+3}\cdots : z_j \in \{0,1,2\}\}, \end{aligned}$$

because such open intervals form a topological basis. Since $U$ contains the open interval

$$\begin{aligned} V &= (0.a_1a_2a_3\cdots a_n1, 0.a_1a_2a_3\cdots a_n2) \\ &= (0.a_1a_2a_3\cdots a_n + 3^{-n-1}, 0.a_1a_2a_3\cdots a_n + 2\cdot 3^{-n-1}), \end{aligned}$$

which is one of the intervals in the complement of $K$, we conclude that $\bar{K}$ does not contain $V$ and therefore does not contain $U$.

Exercise 9.4.1. Prove that $A$ is nowhere dense if and only if $X - A$ contains an open and dense subset.

Exercise 9.4.2. Prove that $A$ is nowhere dense if and only if for any nonempty open subset $U$, there is a nonempty open subset $V \subset U$, such that $A \cap V = \emptyset$.

Exercise 9.4.3. For a *closed* subset $A \subset X$, the following are equivalent.

- $A$ is nowhere dense.
- $A$ has empty interior.
- $X - A$ is dense.

If $A$ is not assumed to be closed, is there any relation between the properties?

Exercise 9.4.4. Prove that a subset of a nowhere dense subset is nowhere dense.

Exercise 9.4.5. Prove that the closure of a nowhere dense subset is nowhere dense.

Exercise 9.4.6. Prove that the union of two nowhere dense subsets is nowhere dense. How about countable union?

Exercise 9.4.7. Let $A \subset Y \subset X$. Prove that if $A$ is a nowhere dense subset of $Y$, then $A$ is a nowhere dense subset of $X$.

Exercise 9.4.8. Let $U$ be an open subset of $X$. Prove that if $A \subset X$ is nowhere dense in $X$, then $A \cap U$ is nowhere dense in $U$.

Note that, combined with Exercise 9.4.7, a subset $A \subset U$ is nowhere dense in $U$ if and only if it is nowhere dense in $X$.

Exercise 9.4.9. Prove that $A \times B$ is nowhere dense in the product space $X \times Y$ if and only if either $A$ or $B$ is nowhere dense.

Exercise 9.4.10. We may extend the construction of Cantor set in the following way: First delete the open middle interval of length $r_1$ from $[0,1]$ to get $K_1$. Next delete the open middle intervals of length $r_2$ from each of the two disjoint intervals that makes $K_2$ to get $K_3$. Keep going and get a sequence $K_n$. Let $K = \cap_n K_n$.

1. Show that $K_n$ consists of $2^n$ disjoint closed intervals of length

$$2^{-n}(1 - r_1 - 2r_2 - 2^2 r_3 - \cdots - 2^{n-1} r_n)$$

each. Therefore the construction can go through and produce $K$ if and only if $\sum_{n=1}^{\infty} 2^{n-1} r_n \leq 1$.

2. Prove that $K$ is nowhere dense.
3. Compute the measure of $K$ and show that the measure can be positive.

The answer to the last question shows that nowhere dense subsets can be large from analysis viewpoint.

In measure theory, a countable union of subsets of measure zero still has measure zero. In topology, although a countable union of nowhere dense subsets may become dense, such subsets should still be considered as small. Indeed, for certain topological spaces, a countable union of nowhere dense subsets cannot be the whole space.

**Theorem 9.4.1** (Baire Category Theorem). *A complete metric space has the following equivalent properties.*

1. *Any countable union of nowhere dense subsets has empty interior. (In particular, the union cannot be the whole space.)*
2. *Any countable union of nowhere dense closed subsets has empty interior.*
3. *If a countable union has nonempty interior, then the closure of some subset in the union has nonempty interior.*
4. *Any countable intersection of open and dense subsets is still dense. (In particular, the intersection is not empty.)*

*A locally compact Hausdorff space also has the properties.*

A Hausdorff space is *locally compact* if any point has a compact neighborhood. The condition is the same as the following: For any $x \in X$, there is an open subset $U$ containing $x$, such that $\bar{U}$ is compact. See Exercise 7.5.29.

Spaces satisfying the properties in Theorem 9.4.1 are called *Baire spaces*. A Baire space cannot be a countable union of closed subsets with empty interior.

*Proof.* First we show the properties are equivalent. The equivalence between the first two properties is due to Exercise 9.4.5. The third property is the contrapositive of the first, because not nowhere dense is the same as the closure having nonempty interior. The fourth property is the translation of the second property by taking the complements. Specifically, the complement of closed is open, the complement of a nowhere dense closed subset is an open and dense subset, and the complement of a subset with empty interior is a dense subset.

Next we prove that a complete metric space $X$ has the fourth property. Let $U_1, U_2, \dots,$ be open and dense subsets. For any $x \in X$ and $\epsilon > 0$, we need to show that $B(x,\epsilon) \cap (\cap_n U_n)$ is not empty.

Since $U_1$ is dense, we can find $a_1 \in B(x,\epsilon) \cap U_1$. Since $U_1$ is open, we may further find a closed ball $\bar{B}(a_1,\delta_1) \subset B(x,\epsilon) \cap U_1$.

Now in place of $B(x,\epsilon)$ and $U_1$, we take $B(a_1,\delta_1)$ and $U_2$. Since $U_2$ is dense and open, we can find a closed ball $\bar{B}(a_2,\delta_2) \subset B(a_1,\delta_1) \cap U_2$. Keep going, we find balls $B(a_n,\delta_n)$, such that

$$\bar{B}(a_{n+1},\delta_{n+1}) \subset B(a_n,\delta_n), \quad \bar{B}(a_n,\delta_n) \subset B(x,\epsilon) \cap U_n.$$

The first property tells us

$$m > n \implies B(a_m,\delta_m) \subset B(a_n,\delta_n) \implies a_m \in B(a_n,\delta_n) \implies d(a_n,a_m) < \delta_n.$$

We may also choose $\delta_n$ to satisfy $\lim \delta_n = 0$. Then $a_n$ is a Cauchy sequence. Since $X$ is complete, the sequence converges to some $a \in X$.

Since $a_m \in B(a_n,\delta_n)$ for $m > n$, we get $a \in \bar{B}(a_n,\delta_n)$ for any $n$. Then by $\bar{B}(a_n,\delta_n) \subset B(x,\epsilon) \cap U_n$, we get $a \in B(x,\epsilon) \cap (\cap_n U_n)$.

Finally, in a locally compact Hausdorff space, we may replace $B(x,\epsilon)$ by an open neighborhood $V$ of $x$ with compact closure. Then by making use of the fact that compact Hausdorff spaces are normal, we can still find open subsets $V_n$, such that

$$\bar{V}_{n+1} \subset V_n, \quad \bar{V}_n \subset V \cap U_n.$$

The compactness of $\bar{V}$ implies that $\cap \bar{V}_n$ is nonempty. Then any point $a \in \cap \bar{V}_n$ lies in $V \cap (\cap_n U_n)$. □

**Example 9.4.3.** Consider $\mathbb{Q}$ with the usual topology. Any single point subset is nowhere dense. However, the countable union of all the single point subsets is actually the whole space. Therefore the usual topology on $\mathbb{Q}$ is not a Baire space and cannot be induced from a complete metric.

**Example 9.4.4.** Consider $\mathbb{R}$ with the usual topology. Any single point subset is nowhere dense. However, the countable union $\mathbb{Q}$ of all the rational single point subsets is dense in the whole space. As expected from the Baire category theorem, the union $\mathbb{Q}$ indeed has empty interior. However, we cannot make the stronger conclusion that the union is nowhere dense. The stronger conclusion would mean that the *closure* of the union has empty interior.

Exercise 9.4.11. Construct an example to show that a Baire space can be a countable union of (not necessarily closed) subsets with empty interior.

Exercise 9.4.12. Prove that any open subset of a Baire space is a Baire space. Moreover, if $X = \cup_i X_i$ is a union of open subsets, then $X$ is a Baire space if and only if each $X_i$ is a Baire space.

Exercise 9.4.13. Suppose a Baire space is a countable union of closed subsets. Prove that the union of the interiors of the closed subsets is dense. Equivalently, if a Baire space is a countable union of some subsets, then the union of the interiors of the closures of the subsets is dense.

Exercise 9.4.14. Let $X$ be a Fréchet Baire space. Let $C$ be a countable subset with empty interior. Prove that $X - C$ is a Baire space. In particular, this shows that the irrational numbers with the usual topology is a Baire space (the fact also follows from Exercise 9.3.12).

Exercise 9.4.15. Let $X$ be a Fréchet Baire space.

1. Prove that a countable intersection of open and dense subsets cannot be a countable subset with empty interior.
2. Suppose further that $X$ has no isolated points (see Example 9.4.1). Prove that a countable intersection of open and dense subsets cannot be a countable subset.

In particular, if all nonempty open subsets are always uncountable, then a countable intersection of open and dense subsets must be uncountable.

Next is an application of the Baire category theorem. In mathematical analysis, you may have constructed functions such as $\sum b^n \cos(a^n \pi x)$, which is continuous but nowhere differentiable whenever $0 < b < 1$ and $a$ is an odd integer satisfying $ab > 1 + \dfrac{3\pi}{2}$. However, by using Theorem 9.4.1, we can show that continuous and nowhere differentiable functions actually appears everywhere.

**Theorem 9.4.2.** *Continuous and nowhere differentiable functions form a dense subset of $C[0,1]$ in the $L_\infty$-metric.*

*Proof.* For any $n$, let $U_n$ consist of those continuous functions $f$, such that for any $s \in [0,1]$, we have

$$|f(s) - f(t)| > n|s - t|$$

for some $t \in [0,1]$. If $f(x)$ is differentiable at $s \in (0,1)$, then there is $\delta > 0$, such that $(s - \delta, s + \delta) \subset [0,1]$, and

$$t \in (s - \delta, s + \delta) \implies \left| \frac{f(s) - f(t)}{s - t} \right| < |f'(s)| + 1.$$

Moreover, if $f$ is bounded by $M$ on $[0,1]$, then we also have

$$t \in [0,1] - (s - \delta, s + \delta) \implies \left| \frac{f(s) - f(t)}{s - t} \right| < \frac{2M}{\delta}.$$

Therefore we have $f \notin U_n$ for any $n$ satisfying $n > |f'(s)| + 1$ and $n > \dfrac{2M}{\delta}$. Similar argument can be made in case $f$ is right differentiable at $s = 0$ and in case $f$ is left differentiable at $s = 1$. This shows that functions in $\cap U_n$ are nowhere differentiable.

We claim that $U_n$ is open and dense in the $L_\infty$-metric. Then by the completeness of the $L_\infty$-metric on $C[0,1]$ (see Example 9.3.2) and Theorem 9.4.1, the countable intersection $\cap U_n$ is also $L_\infty$-dense in $C[0,1]$.

Now we verify that $U_n$ is $L_\infty$-open. Fix $f \in U_n$. For any $s \in [0,1]$, we have $|f(s) - f(t)| > n|s - t|$ for some $t \in [0,1]$. By the continuity of $f$, there is $\delta_s > 0$, such that

$$u \in [s - \delta_s, s + \delta_s] \cap [0,1] \implies |f(u) - f(t)| > n|u - t|.$$

Since continuous functions reach minimum on bounded closed intervals, we have

$$\epsilon_s = \frac{1}{2} \min\{|f(u) - f(t)| - n|u - t| \colon u \in [s - \delta_s, s + \delta_s] \cap [0,1]\} > 0.$$

Then for $g \in B_\infty(f, \epsilon_s)$ and $u \in [s - \delta_s, s + \delta_s] \cap [0,1]$, we have

$$|g(u) - g(t)| \geq |f(u) - f(t)| - 2d_\infty(f, g) > |f(u) - f(t)| - 2\epsilon_s \geq n|u - t|.$$

The compact interval $[0,1]$ may be covered by finitely many intervals $(s_1 - \delta_{s_1}, s_1 + \delta_{s_1})$, $(s_2 - \delta_{s_2}, s_2 + \delta_{s_2})$, $\dots$, $(s_k - \delta_{s_k}, s_k + \delta_{s_k})$. Then for $\epsilon = \min\{\epsilon_{s_1}, \epsilon_{s_2}, \dots, \epsilon_{s_k}\}$, the estimation on $|g(u) - g(t)|$ above implies that $B_\infty(f, \epsilon) \subset U_n$. This proves that $U_n$ is $L_\infty$-open.

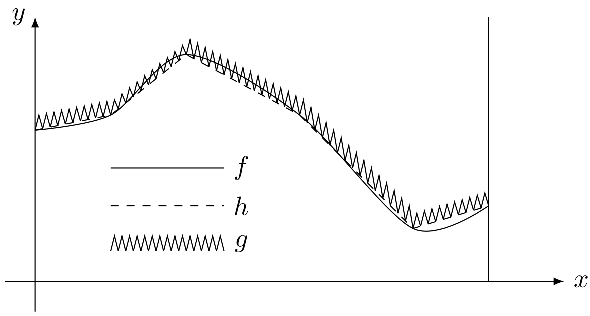



**Figure 9.4.2.** *$L_\infty$-approximation by sawtooth function*

Finally, we verify that $U_n$ is $L_\infty$-dense. This means that for any continuous function $f$ and any $\epsilon > 0$, we need to find $g \in U_n$, such that $d_\infty(f, g) < \epsilon$. Since $f$ is continuous (and therefore uniformly continuous), we may find a piecewise linear continuous function $h$ satisfying $d_\infty(f, h) < \dfrac{\epsilon}{2}$. Such function can be constructed, for example, by choosing a sufficiently refined partition $a = t_0 < t_1 < \cdots < t_k = b$ and connect the point $(t_i, f(t_i))$ in $\mathbb{R}^2$ to the point $(t_{i+1}, f(t_{i+1}))$ by straight line. It is easy to see that the piecewise linear function $h$ is Lipschitz: $|h(t) - h(s)| \leq M|t - s|$ for a constant $M$ and all $s, t \in [0,1]$. Now we construct a continuous "sawtooth function" $\sigma(t)$, such that $|\sigma(t)| < \frac{\epsilon}{2}$ for any $t \in [0,1]$, and for any $s$, there is $t$, such that $|\sigma(s) - \sigma(t)| > (n + M)|s - t|$. This implies that for $g = h + \sigma$, we have

$$d_\infty(f, g) \le d_\infty(f, h) + d_\infty(h, g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and for any $s$, there is $t$, such that

$$|g(s) - g(t)| \ge |\sigma(s) - \sigma(t)| - |h(s) - h(t)| > n|s - t|.$$

In particular, we have $g \in U_n$. □

Exercise 9.4.16. Let $U_n$ consist of continuous functions $f$ on $[0,1]$ with the property that for any $t$, there is $s$, such that $|s-t| < \frac{1}{n}$ and $\left|\frac{f(s) - f(t)}{s - t}\right| > n$. Prove that $U_n$ is open and dense in the $L_\infty$-metric. Moreover, prove that functions in $\cap U_n$ are nowhere differentiable. This leads to another proof of Theorem 9.4.2.

Exercise 9.4.17. Suppose $X$ is a Baire space and $\{f_i\}_{i \in I}$ is a collection of functions on $X$. Suppose the collection is pointwise uniformly bounded: For any $x \in X$, there is $M_x$, such that $|f_i(x)| < M_x$ for all $i \in I$. Prove that the collection is uniformly bounded on a nonempty open subset $U$: There is $M$, such that $|f_i(x)| < M$ for all $i \in I$ and all $x \in U$.

Exercise 9.4.18. Suppose $f_n$ is a sequence of continuous functions on $[0,1]$ pointwise converging to $f$. For each $\epsilon > 0$ and natural number $N$, let

$$U_N(\epsilon) = \{t \colon |f_m(t) - f_n(t)| \le \epsilon \text{ for all } m, n > N\}.$$

Let $V(\epsilon)$ be the union of the interior of $U_N(\epsilon)$ for all $N$. Prove that $f$ is continuous on a dense subset of $[0,1]$ in the following steps.

1. Prove that for any fixed $\epsilon$, we have $C[0,1] = \cup_N U_N(\epsilon)$.
2. Use Exercise 9.4.13 to prove that $V(\epsilon)$ is dense.
3. Prove that $A = \cap_k V\left(\frac{1}{k}\right)$ is dense.
4. Prove that $f$ is continuous at every point in $A$.

Note that $[0,1]$ can be replaced by any Baire space, and the functions can be replaced by maps into a metric space. Moreover, the subset $A$ is not countable by Exercise 9.4.15.

## 9.5 Infinite Product

The *product* $\times_{i \in I} X_i$ of a collection of sets $X_i$, $i \in I$, consists of elements of the form

$$(x_i)_{i \in I}, \quad x_i \in X_i.$$

In other words, an element of the product is a choice of one element from each set in the collection. For each fixed index $j$, the projection map

$$\pi_j \colon \times_{i \in I} X_i \to X_j,\ (x_i)_{i \in I} \mapsto x_j$$

simply means the choice of the element in $X_j$.

When the sets $X_i$ are the copies of the same set $X$, the product $\times_{i\in I}X_i$ is also denoted $X^I$, which is simply the set of all maps $I \to X$.

**Definition 9.5.1.** The *product topology* on the cartesian product $\times_{i\in I}X_i$ is the coarsest topolgy such that all the projections $\pi_j$ are continuous.

The continuity of the projection $\pi_j$ means that the subset

$$\pi_j^{-1}(U) = U \times (\times_{i\in I, i\neq j}X_i)$$

is open for any open subset $U \subset X_j$. By taking finite intersections of these, the product topology has a topological basis given by $\times_{i\in I}U_i$, where $U_i \subset X_i$ are open, and $U_i = X_i$ for all but finitely many $i$. More specifically, there is a finite subset $K \subset I$ of indices, such that $U_i = X_i$ for $i \notin K$. Then

$$\times_{i\in I}U_i = U_K \times (\times_{i\notin K}X_i), \quad U_K = \times_{i\in K}U_i, \quad U_i \subset X_i \text{ open.}$$

From this description, it is easy to see that the product topology is also induced by a topological basis of the form $U \times (\times_{i\notin K}X_i)$, where $K$ is any finite subset of $I$, and $U$ is any open subset of $\times_{i\in K}X_i$.

**Example 9.5.1.** The space of infinite sequences of 0s and 1s is

$$X = \{(a_1a_2a_3\cdots) : a_i = 0 \text{ or } 1\}.$$

The space is the product of countably many copies of $\{0, 1\}$. If $\{0, 1\}$ has the discrete topology, then a topological basis of the product topology is given by the subsets

$$X_{c_1c_2\cdots c_n} = \{(c_1c_2\cdots c_na_{n+1}a_{n+2}\cdots) : a_i = 0 \text{ or } 1 \text{ for } i > n\}$$

of all sequences with the same specific beginning terms. The topology is called the *cylinder topology*.

**Example 9.5.2.** The product of $I$ copies of $\mathbb{R}$ is $\mathbb{R}^I$. An element of $\mathbb{R}^I$ can be considered as a real number valued function $f : I \to \mathbb{R}$. For example, $\mathbb{R}^{[0,1]}$ is the collection of all the functions on $[0, 1]$, which contains all the continuous functions $C[0, 1]$.

Given the usual topology on $\mathbb{R}$, the product topology on $\mathbb{R}^I$ is given by the topological basis

$$B((a_1, b_1), \dots, (a_n, b_n), i_1, \dots, i_n) = \{f : I \to \mathbb{R}, a_1 < f(i_1) < b_1, \dots, a_n < f(i_n) < b_n\}.$$

Equivalently, we can also use the topological basis

$$B(a_1, \dots, a_n, i_1, \dots, i_n, \epsilon) = \{f : I \to \mathbb{R}, |f(i_1) - a_1| < \epsilon, \dots, |f(i_n) - a_n| < \epsilon\},$$

which shows that the product topology is exactly the pointwise convergence topology.

Exercise 9.5.1. Given a topological basis $\mathcal{B}_i$ for each $X_i$, describe a topological basis of the product topology. Do you have a similar description for the topological subbasis?

Exercise 9.5.2. Suppose the index set $I$ is a disjoint union of subsets $I'$ and $I''$. Then we have $\times_{i\in I}X_i = (\times_{i\in I'}X_i)\times(\times_{i\in I''}X_i)$. Prove that the product topology on $\times_{i\in I}X_i$ is the product of the product topologies on $\times_{i\in I'}X_i$ and $\times_{i\in I''}X_i$. Moreover, extend the property to the decomposition of $I$ into a disjoint union of more than two (perhaps infinitely many) subsets.

Exercise 9.5.3. Let $A_i \subset X_i$ be a subset. Then the subset $\times_{i\in I}A_i \subset \times_{i\in I}X_i$ has one topology given by the subspace of the product of the topologies of $X_i$, and another topology given by the product of the subspace topologies of $A_i$. Compare the two topologies.

Exercise 9.5.4. Let $A_i \subset X_i$ be a subset. Prove that $\overline{\times_{i\in I}A_i} = \times_{i\in I}\bar{A}_i$. What about the limit points of the product?

Exercise 9.5.5. Let $X_i$ be nonempty. Prove that the product $\times_{i\in I}X_i$ is Hausdorff if and only if each $X_i$ is Hausdorff.

Exercise 9.5.6. Let $X_i$ be nonempty. Prove that the product $\times_{i\in I}X_i$ is regular if and only if each $X_i$ is regular.

Exercise 9.5.7. Let $X_i$ be nonempty. Prove that the product $\times_{i\in I}X_i$ is connected if and only if each $X_i$ is connected. For the sufficiency, you may fix $(x_i) \in \times_{i\in I}X_i$ and prove that

$$\cup_{K\subset I, K\text{ finite}}(x_i)_{i\notin K}\times(\times_{i\in K}X_i)$$

is a dense subset of $\times_{i\in I}X_i$.

Exercise 9.5.8. A map $f\colon Y \to \times_{i\in I}X_i$ is given by maps $f_i\colon Y \to X_i$. Prove that $f$ is continuous if and only if each $f_i$ is continuous.

Exercise 9.5.9. Let $X_i$ be nonempty. Prove that the product $\times_{i\in I}X_i$ is path connected if and only if each $X_i$ is path connected.

Exercise 9.5.10. Given maps $f_i\colon X_i \to Y_i$, we get a map $\times_{i\in I}f_i\colon \times_{i\in I}X_i \to \times_{i\in I}Y_i$ of the products. Prove that $\times_{i\in I}f_i$ is continuous if and only if each $f_i$ is continuous.

Exercise 9.5.11. Let $Y$ be a set, let $X_i$, $i \in I$, be topological spaces, and let $f_i\colon Y \to X_i$ be maps. Then we have the coarsest topology $\mathcal{T}$ on $Y$, such that all $f_i$ are continuous.

1. Prove that $g\colon Z \to Y$ is continuous if and only if $f_i \circ g\colon Z \to X_i$ are continuous.
2. Prove that the product map $\times_{i\in I}f_i\colon Y \to \times_{i\in I}X_i$ maps open subsets to open subsets of the image subspace.

Exercise 9.5.12. Prove that all subsets of the form $\times_{i\in I}U_i$, with $U_i \subset X_i$ being open (and without assuming $U_i = X_i$ for all all but finitely many $i$), form a topological basis. The product spaces in the following questions have the topology induced by this topological basis. Moreover, the real line $\mathbb{R}$ has the usual topology.

1. Show that the induced topology on the product $\mathbb{R}^{\mathbb{N}}$ is strictly finer than the product topology.

2. Show that the diagonal map $\mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, $x \mapsto (x, x, x, \dots)$ is not continuous with respect to the induced topology on $\mathbb{R}^{\mathbb{N}}$.
3. Prove that if any point in $X$ has a minimal open subset containing it (see Exercise 7.3.15), then the diagonal map $X \to X^I$ is continuous with respect to the induced topology on $X^I$.

**Theorem 9.5.2** (Tychonoff[30] Theorem). *Suppose $X_i$ are nonempty topological spaces. Then the product topology on $\times_{i\in I} X_i$ is compact if and only if each $X_i$ is compact.*

In Theorem 7.5.2, we already know the product of two (as well as finitely many) compact spaces is compact. However, the same property for the infinite product is a much deeper one. Notice that in Exercise 9.5.7, the connected property is extended by considering the infinite product as some "limit" (i.e., the closure) of all the finite products. The same principle may be used for proving the compact property. However, we cannot simply take the union of all the finite products because an infinite union of compact subsets may not be compact. A much more refined setup has to be arranged.

A key step in the proof of the sixth property in Theorem 7.5.2 is the following: Suppose $\mathcal{U} = \{U_i \times V_i\}$ is a "rectangular" open cover of $X \times Y$, such that the "slice" $x \times Y$ can be covered by finitely many subsets from $\mathcal{U}$. Then there is an open neighborhood $U_x$ of $x$, such that $U_x \times Y$ can also be covered by the same finitely many subsets from $\mathcal{U}$. Now if $X$ is compact, then the open cover $\{U_x : x \in X\}$ has a finite subcover, and this implies that $X \times Y$ is covered by finitely many from $\mathcal{U}$. The contrapositive of the key step is summarized in the following technical result.

**Lemma 9.5.3.** *Suppose $X$ is compact. Suppose $\mathcal{U} = \{U_i \times V_i\}$ is an open cover of $X \times Y$ by subsets in the product topological basis. If $\mathcal{U}$ has no finite subcover, then there is $x \in X$, such that $x \times Y$ cannot be covered by finitely many subsets from $\mathcal{U}$.*

For compact $X_i$, the lemma can be used to prove the compactness of a finite product $X_1 \times X_2 \times \cdots \times X_n$ as follows. If $\mathcal{U}$ is an open rectangular cover of the product with no finite subcover, by the lemma and the compactness of $X_1$, there is $x_1 \in X_1$, such that $x_1 \times X_2 \times \cdots \times X_n$ cannot be covered by finitely many from $\mathcal{U}$. Next, we wish to apply the lemma to the open rectangular cover $\mathcal{U}$ of $x_1 \times X_2 \times \cdots \times X_n$ and use the compactness of $X_2$ to conclude that there is $x_2 \in X_2$, such that $x_1 \times x_2 \times X_3 \times \cdots \times X_n$ cannot be covered by finitely many from $\mathcal{U}$. However, the collection $\mathcal{U}$ is not composed of the open subsets of $X_2 \times \cdots \times X_n$, so that the lemma cannot be directly applied. The technical lemma has to be modified, and modification can be similarly proved.

**Lemma 9.5.4.** *Suppose $X$ is compact. Suppose $\mathcal{U} = \{R_i \times U_i \times V_i\}$ is a collection of open subsets in the product topological basis of $W \times X \times Y$. If $w \times X \times Y$ is*

[30] Andrey Nikolayevich Tychonoff, born October 30, 1906 in Gzhatsk, Russia, died November 8, 1993 in Moscow, Russia. Tychonoff made important contributions to topology, functional analysis and mathematical physics. A regularization method named after him is the most commonly used method of treating ill-posed problems.

*covered by $\mathcal{U}$ but cannot be covered by finitely many subsets from $\mathcal{U}$, then there is $x \in X$, such that $w \times x \times Y$ cannot be covered by finitely many subsets from $\mathcal{U}$.*

Using the modified lemma, the argument can continue by induction. We successively find $x_1, x_2, \ldots, x_n$, such that $x_1 \times \cdots \times x_i \times X_{i+1} \times \cdots \times X_n$ cannot be covered by finitely many subsets from $\mathcal{U}$. Of course, at the end, we get the contradiction that the single point $x_1 \times \cdots \times x_n = (x_1, \ldots, x_n)$ cannot be covered by finitely many subsets from $\mathcal{U}$.

To carry out the similar argument for an infinite product, we first need to arrange the index set in order and carry out certain *transfinite induction*. This requires the following axiom.

**Theorem 9.5.5** (Well Ordering Theorem). *Any set $I$ can be given an order $<$, such that any subset has a minimum element.*

An *order* is a relation $i < j$ (also written as $j > i$) satisfying the following properties:

- *trichotomy*: For any $i$, $j$, one and only one of the relations $i < j$, $i = j$, $i > j$ must hold.
- *transitivity*: If $i < j$ and $j < k$, then $i < k$.

We also use $i \leq j$ (also written as $j \geq i$) to denote $i < j$ or $i = j$. The minimum $k = \min J$ of a subset $J \subset I$ satisfies $k \in J$ and $k \leq j$ for any $j \in J$. If any subset has a minimum element, as is the case in the theorem above, then we say the set is *well ordered*.

The well ordering theorem is an axiom because it has to be postulated in the set theory. It is a theorem because it is equivalent to some other fundamental axioms in the set theory, in particular the *axiom of choice* and *Zorn's lemma*.

Now we are ready to prove the compactness of the infinite product.

*Proof of Theorem 9.5.2.* Since each space $X_i$ is the image of the product space under the continuous projection map, by the sixth property in Theorem 7.5.2, the compactness of the product space implies the compactness of each space $X_i$.

Conversely, we assume each $X_i$ is compact and prove that the product is compact. We make the index set $I$ well ordered. Assume $\mathcal{U}$ is an open cover of the product space with no finite subcover. We try to successively find $x_i \in X_i$, such that for any $j \in I$, the product

$$Y_j = (x_i)_{i \leq j} \times (\times_{i > j} X_i)$$

cannot be covered by finitely many subsets from $\mathcal{U}$.

By Exercise 7.5.7, we may further assume that the subsets in $\mathcal{U}$ are from the product topological basis. In other words, they are of the form $U_K \times (\times_{i \notin K} X_i)$, $U_K = \times_{i \in K} U_i$, for some finite $K \subset I$. The assumption allows us to apply Lemma 9.5.3 and Lemma 9.5.4.

Since $I$ is well ordered, it has a minimum element $j_0$, the "initial index". Since $\mathcal{U}$ is an open cover of $\times_{i\in I}X_i = X_{j_0} \times (\times_{i>j_0}X_i)$ with no finite subcover, and $X_{j_0}$ is compact, by Lemma 9.5.3, there is $x_{j_0} \in X_{j_0}$, such that $Y_{j_0} = x_{j_0} \times (\times_{i>j_0}X_i)$ cannot be covered by finitely many subsets from $\mathcal{U}$.

Now suppose for an index $j$, we have already found $x_i \in X_i$ for all $i < j$, such that $Y_i$ cannot be covered by finitely many subsets from $\mathcal{U}$ for any $i < j$. We claim that

$$\cap_{i<j}Y_i = (x_i)_{i<j} \times (\times_{i\geq j}X_i)$$

cannot be covered by finitely many subsets from $\mathcal{U}$. Once we know the claim, then we may apply Lemma 9.5.4 to

$$W = \times_{i<j}X_i, \quad w = (x_i)_{i<j}, \quad X = X_j, \quad Y = \times_{i>j}X_i, \quad w \times X \times Y = \cap_{i<j}Y_i,$$

and get $x_j \in X_j$, such that

$$w \times x_j \times Y = (x_i)_{i<j} \times x_j \times (\times_{i>j}X_i) = Y_j$$

cannot be covered by finitely many subsets from $\mathcal{U}$.

The claim can be proved by contradiction. Assume $\cap_{i<j}Y_i$ can be covered by finitely many subsets from $\mathcal{U}$, which we have assumed to be of the form

$$U_{K_1} \times (\times_{i\notin K_1}X_i), \quad U_{K_2} \times (\times_{i\notin K_2}X_i), \quad \dots, \quad U_{K_n} \times (\times_{i\notin K_n}X_i),$$

where $K_1, K_2, \dots, K_n$ are finite subsets of $I$. The union $K = K_1 \cup K_2 \cup \cdots \cup K_n$ is still finite, and the fact about the finite cover is independent of the factor spaces $X_i$ with index $i \notin K$. In other words, the finite cover assumption is equivalent to that the projection

$$\pi_K(\cap_{i<j}Y_i) = (x_i)_{i<j, i\in K} \times (\times_{i\geq j, i\in K}X_i)$$

is covered by the projections of the finitely many subsets in $\mathcal{U}$

$$\pi_K\left(U_{K_k} \times (\times_{i\notin K_k}X_i)\right) = U_{K_k} \times (\times_{i\in K-K_k}X_i), \quad k = 1, 2\dots, n.$$

Let $i_0 = \max\{i \in K : i < j\}$. Then the fact about the finite cover of the projections imply that $Y_{i_0}$ is also covered by the original finitely many subsets $U_{K_k} \times (\times_{i\notin K_k}X_i)$ from $\mathcal{U}$. Since $i_0 < j$, this contradicts with the "inductive" assumption that $Y_i$ cannot be covered by finitely many subsets from $\mathcal{U}$ for any $i < j$.

Now we have found (via transfinite induction) $x_i \in X_i$ for each $i \in I$, such that $Y_i$ cannot be covered by finitely many subsets from $\mathcal{U}$ for any $i$. If the order we put on $I$ has a maximal element $j_1$, the "terminal index", then $Y_{j_1} = (x_i)_{i\leq j_1} = (x_i)_{i\in I}$ is a single element of the product space. Therefore $Y_{j_1}$ can be covered by one subset from $\mathcal{U}$, and we get a contradiction.

It is not difficult to impose an order on $I$ so that it has a maximal element. Basically we may delete a point $\bar{\imath}$ from $I$ and make $I - \bar{\imath}$ into a well ordered set. Then by defining $\bar{\imath} > i$ for any $i \neq \bar{\imath}$, the index set $I$ becomes a well ordered set with $\bar{\imath}$ as the maximal element. $\square$

Exercise 9.5.13. Prove Lemma 9.5.4 by showing that if $w \times x \times Y$ can be covered by finitely many subsets from $\mathcal{U}$, then there is an open subset $U \subset X$, such that $x \in U$ and $w \times U \times Y$ can be covered by finitely many subsets from $\mathcal{U}$.

Exercise 9.5.14. Consider a sequence of sequences

$$\{x_n^{(1)} : n \in \mathbb{N}\}, \quad \{x_n^{(2)} : n \in \mathbb{N}\}, \quad \{x_n^{(3)} : n \in \mathbb{N}\}, \quad \dots .$$

Explain that if each sequence is a subsequence of the previous one, then $\{x_n^{(n)} : n \in \mathbb{N}\}$ is essentially (meaning with the exception of finitely many terms) a subsequence of each sequence. Then use the fact to prove that a countable product of sequentially compact spaces (see Exercise 7.6.16) is sequentially compact.

Finally, we consider the product of metric spaces. Suppose $X_i$ has metric $d_i$, such that the diameters of $X_i$ are uniformly bounded. In other words, there is a constant $M$ (independent of $i$), such that for any $i$ and $x, y \in X_i$, we have $d_i(x, y) \le M$. Then we may introduce the $L_\infty$-metric on the product $\times_{i\in I} X_i$

$$d((x_i), (y_i)) = \sup_{i\in I} d_i(x_i, y_i).$$

Clearly, the balls in this metric are

$$B((x_i), \epsilon) = \times_{i\in I} B_{d_i}(x_i, \epsilon).$$

This implies the following result on the topology induced by the metric.

**Proposition 9.5.6.** *Suppose $X_i$ are nonempty metric spaces with uniformly bounded diameter. Then the topology induced by the $L_\infty$-metric on the product $\times_{i\in I} X_i$ is finer than the product topology. Moreover, the topology is the same as the product topology if and only if for any $\epsilon > 0$, there are only finitely many $X_i$ with diameter $> \epsilon$.*

In case the product metric topology is the same as the product topology, there are only countably many $X_i$ with positive diameter. Therefore only countably many $X_i$ are allowed to contain more than one points (which are the metric spaces of diameter 0).

In general, the diameters of the factor spaces may not be uniformly bounded, and some factor spaces may even have infinite diameter. However, the topologies are not changed if the metrics $d_i$ are replaced by $\bar{d}_i = \min\{d_i, 1\}$, and we can form the corresponding $L_\infty$-metric by making use of $\bar{d}_i$. In case $I$ is countable, which is essentially necessary if we want the $L_\infty$-metric to induce the product metric, we may embed $I$ into $\mathbb{N}$ and introduce a modified $L_\infty$-metric

$$\bar{d}((x_i), (y_i)) = \sup_{i\in I} \frac{\bar{d}_i(x_i, y_i)}{i},$$

which induces the product topology. We remark that if the diameters are uniformly bounded, then we may also use

$$\bar{d}((x_i), (y_i)) = \sup_{i\in I} \frac{d_i(x_i, y_i)}{i}$$

to induce the product topology on the product space.

Exercise 9.5.15. A sequence in the product space $\times_{i\in I}X_i$ is of the form

$$x^n = (x_i^n)_{i\in I}$$

and can be considered as given by one sequence $x_i^1, x_i^2, \dots$ in each $X_i$. Let the product $\times_{i\in I}X_i$ have the $L_\infty$-metric (modified if the diameters are not uniformly bounded).

1. Prove that $\lim_{n\to\infty} x^n = a = (a_i)_{i\in I}$ implies $\lim_{n\to\infty} x_i^n = a_i$.
2. Prove that if the $L_\infty$-metric induces the product topology, then $\lim_{n\to\infty} x_i^n = a_i$ for each $i$ implies $\lim_{n\to\infty} x^n = a = (a_i)_{i\in I}$.

Exercise 9.5.16. Suppose a product of metric spaces has the $L_\infty$-metric.

1. Prove that if a sequence in the product space is Cauchy, then each component sequence is also Cauchy.
2. Prove that if the $L_\infty$-metric induces the product topology, and each component sequence is Cauchy, then the sequence in the product space is also Cauchy.
3. Prove that if the $L_\infty$-metric induces the product topology, then the $L_\infty$-metric is complete if and only if each factor space is complete.
4. Discuss the relation between the completeness of the product space and each factor space in general.

Exercise 9.5.17. Suppose $X_i$, $i \in \mathbb{N}$, are metric spaces with uniformly bounded diameters.

1. Prove that
$$d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{d_i(x_i,y_i)}{2^i}$$
is a metric on $\times_{i\in I}X_i$.
2. Prove that $d$ induces the product topology.
3. Can $d$ and the $L_\infty$-metric be compared?
4. Does $d$ have the similar properties as the $L_\infty$-metric?

## 9.6 Space-Filling Curve

In 1877, Cantor proved that the interval $[0,1]$ and the square $[0,1]^2$ contain the same number of points. In other words, there is a one-to-one correspondence between the two sets. The astonishing discovery (Cantor said: "I see it, but I don't believe it!") showed that the dimension is not a set-theoretical concept. The difference in dimensions has to involve topology. Then in 1890, Peano[31] showed that there is a surjective continuous map from the interval to the square. In other words, there is a (square) *space filling curve*. Note that the interval and the square are not homeomorphic (by an argument similar to Example 7.3.7, for example). Therefore the fourth property in Theorem 7.5.2 implies that the space filling curve cannot be one-to-one.

[31]Giuseppe Peano, born August 27, 1858 in Spinetta, Italy, died April 20, 1932 in Turin, Italy. Peano did pioneering work in symbolic logic and made important contributions to mathematical analysis. The famous Peano axiom of natural numbers appeared in his "Arithmetices principia".

**Theorem 9.6.1.** *There is a surjective continuous map* $f\colon [0,1] \to [0,1]^2$.

*Proof.* The curve is constructed as the limit of a sequence of "denser and denser" curves. We choose one edge of the square as the starting curve. Figure 9.6.1 shows the bottom edge curve.

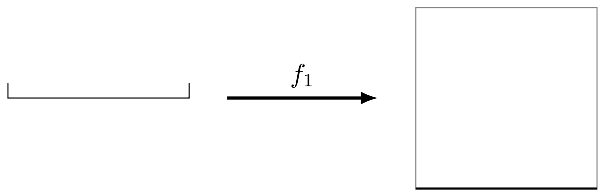



**Figure 9.6.1.** *the starting curve*

For the next curve, we divide the interval and the square into four equal parts, and find a curve that sends the four subintervals to the edges of the four subsquares in a continuous way. Figure 9.6.2 shows such a curve.

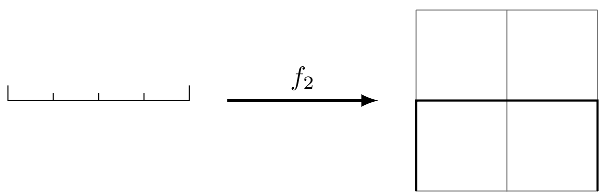



**Figure 9.6.2.** *the next curve*

The process is repeated in the subsequent constructions. Figure 9.6.3 is the third curve. Note that we may consider the transition from $f_1$ to $f_2$ as a process. The third curve $f_3$ is actually obtained by applying the process to each of the four squares in the definition of $f_2$. Of course in applying the process, we need to consider whether the starting segment is the bottom, top, left, or right edge of the square. The result is that both the interval $[0,1]$ and the square $[0,1]^2$ are divided into $4^2$ equal parts, and the curve $f_3$ sends the $4^2$ subintervals to the edges of the $4^2$ subsquares in a continuous way.

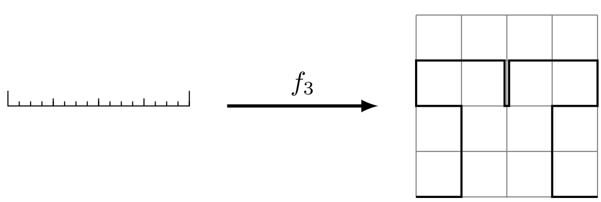



**Figure 9.6.3.** *the third curve*

Figure 9.6.4 is the fourth curve obtained by applying the transition process to

$f_3$. The result is that both the interval $[0,1]$ and the square $[0,1]^2$ are divided into $4^3$ equal parts, and the curve $f_4$ sends the $4^3$ subintervals to the edges of the $4^3$ subsquares in a continuous way. In general, the map $f_{n+1}$ is obtained by applying the transition process to each of the $4^{n-1}$ parts of the map $f_n$. The result is that both the interval $[0,1]$ and the square $[0,1]^2$ are divided into $4^n$ equal parts, and the curve $f_{n+1}$ sends the $4^n$ subintervals to the edges of the $4^n$ subsquares in a continuous way.

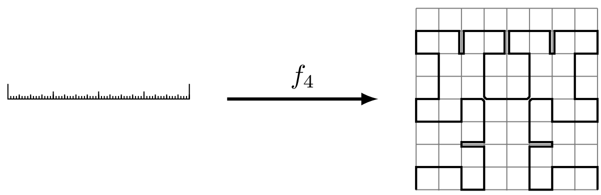



**Figure 9.6.4.** *the fourth curve*

Thus we have a sequence $P_n$ of partitions of $[0,1]$ and a sequence $Q_n$ of partitions of $[0,1]^2$. Moreover, we have maps $\alpha_n\colon P_n \to Q_n$ between the pieces in the partitions. We also have continuous maps $f_n\colon [0,1] \to [0,1]^2$. These data have the following key properties:

1. $P_{n+1}$ is a refinement of $P_n$. $Q_{n+1}$ is a refinement of $Q_n$.
2. $\alpha_n$ is a one-to-one correspondence between the pieces in $P_n$ and the pieces in $Q_n$.
3. If $I' \in P_{n+1}$ is a subset of $I \in P_n$, then $\alpha_{n+1}(I') \in Q_{n+1}$ is a subset of $\alpha_n(I) \in Q_n$.
4. $f_n$ maps $I \in P_n$ into $\alpha_n(I) \in Q_n$.
5. $\lim_{n\to\infty} \|P_n\| = 0$ and $\lim_{n\to\infty} \|Q_n\| = 0$. Here $\|P\| = \max\{\text{diameter}(I)\colon I \subset P\}$ is the size of a partition $P$.

We claim that $f = \lim_{n\to\infty} f_n$ converges and is a surjective continuous map.

For any $t \in [0,1]$ and $m \geq n$, we have $t \in I'$ for some interval $I' \in P_m$ and $t \in I$ for some $I \in P_n$. By the first and the third properties, we have $I' \subset I$ and $\alpha_m(I') \subset \alpha_n(I)$. By the fourth property, we have $f_n(t) \in \alpha_n(I)$ and $f_m(t) \in \alpha_m(I') \subset \alpha_n(I)$. Therefore

$$\|f_m(t) - f_n(t)\| \leq \text{diameter}(\alpha_n(I)) \leq \|Q_n\|.$$

Since $\lim_{n\to\infty} \|Q_n\| = 0$, the sequence $f_n$ satisfies the uniform Cauchy criterion. From calculus, we know that the sequence of continuous maps converges to a continuous map $f$. We note that, by taking $m \to \infty$, the inequality above also implies

$$\|f(t) - f_n(t)\| \leq \|Q_n\|$$

for any $t \in [0,1]$.

It remains to prove that $f$ is surjective. Let $p \in [0,1]^2$ be a point. By the first property, we can find $J_n \in Q_n$, such that $p \in J_n$ and $J_{n+1} \subset J_n$. By the second property, we have $J_n = \alpha_n(I_n)$ for some $I_n \in P_n$. By the third property, we also know $I_{n+1} \subset I_n$. Choose $t_n \in I_n$. Then for $m \geq n$, we have $t_m \in I_m \subset I_n$, so that

$$|t_m - t_n| \leq \text{diameter}(I_n) \leq \|P_n\|.$$

Since $\lim_{n\to\infty} \|P_n\| = 0$, the sequence $t_n$ satisfies the Cauchy criterion and converges to some $t \in [0,1]$. Then we have

$$\|f(t) - f_n(t_n)\| \leq \|f(t) - f(t_n)\| + \|f(t_n) - f_n(t_n)\| \leq \|f(t) - f(t_n)\| + \|Q_n\|.$$

By the continuity of $f$ and $\lim_{n\to\infty} \|Q_n\| = 0$, we conclude that

$$f(t) = \lim_{n\to\infty} f_n(t_n).$$

On the other hand, since $p \in J_n$ and $f_n(t_n) \in \alpha_n(I_n) \in J_n$, we have

$$\|p - f_n(t_n)\| \leq \text{diameter}(\alpha_n(I_n)) \leq \|Q_n\|.$$

By $\lim_{n\to\infty} \|Q_n\| = 0$, this implies that

$$p = \lim_{n\to\infty} f_n(t_n).$$

We conclude that $p = f(t)$. $\square$

Note that the five properties in the proof imply the convergence of the sequence of maps, the continuity of the limit map, and the surjectivity of the limit map. The specific way of constructing the partitions and the maps is not important. The key is that the interval and the square have matching sequences of more and more refined partitions, and continuous maps can be found to preserve the correspondence between the partitions. There are many constructions satisfying the properties. Figure 9.6.5 is Peano's origional construction.

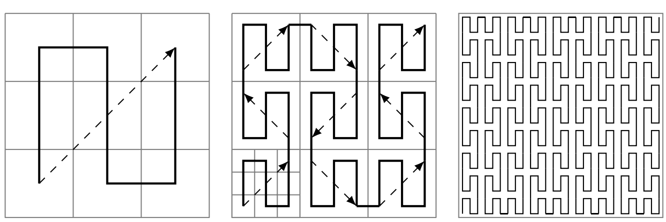


**Figure 9.6.5.** *Peano's construction*

Exercise 9.6.1. Is there a continuous surjective map from $[0,1]$ to $(0,1)^2$? Is there a continuous surjective map from $(0,1)$ to $[0,1]^2$?

Exercise 9.6.2. Show that there is a continuous surjective map from $[0,1]$ to $[0,1]^n$. Show that there is a continuous surjective map from $\mathbb{R}$ to $\mathbb{R}^n$.

Exercise 9.6.3. Explain that the space-filling curve constructed in Theorem 9.6.1 actually does not depend on the specific choice of $f_n$. In other words, the curve is completely determined by $P_n$, $Q_n$ and $\alpha_n$ only.

Exercise 9.6.4. Let the space-filling curve constructed in Theorem 9.6.1 be $f = (\phi, \psi)$. Let $f_n = (\phi_n, \psi_n)$.

1. Let $I \in P_n$, $I' \subset P_{n+2}$, and $I' \subset I$. Prove that there is another $I'' \subset P_{n+2}$, such that the squares $\alpha_{n+2}(I')$ and $\alpha_{n+2}(I'')$ in $Q_n$ are separated by distance $2^{-n-2}$ in both vertical and horizontal directions: For any $(x_1, y_1) \in I'$ and $(x_2, y_2) \in I''$, we have $|x_1 - x_2| \geq 2^{-n-2}$ and $|y_1 - y_2| \geq 2^{-n-2}$.
2. For any $t \in [0,1]$ and $n$. Prove that there is $t_n$, such that $|t - t_n| \leq 4^{-n}$, $|\phi_m(t) - \phi_m(t_n)| \geq 2^{-n-2}$ and $|\psi_m(t) - \psi_m(t_n)| \geq 2^{-n-2}$ for any $m \geq n$.
3. Prove that the coordinate functions of $f$ are not differentiable everywhere.

## 9.7 Space of Maps

Let $\mathcal{F}(X,Y)$ be the set of maps from $X$ to $Y$. Then $\mathcal{F}(X,Y)$ can be identified with the cartesian product $Y^X$ of $X$ copies of $Y$. If $Y$ is a topological space, then we may introduce the *product topology* on $\mathcal{F}(X,Y) = Y^X$. Note that the product topology on $\mathcal{F}(X,Y)$ does not require $X$ to be a topological space. Moreover, Example 9.5.2 shows that the product topology on $\mathcal{F}(X,\mathbb{R})$ is simply the pointwise convergence topology.

On the other hand, for topological spaces $X$ and $Y$, we naturally wish to put a topology on $\mathcal{F}(X,Y)$ so that the evaluation map

$$E(f,x) = f(x) \colon \mathcal{F}(X,Y) \times X \to Y$$

is continuous. If a topology on $\mathcal{F}(X,Y)$ has the property, then the finer topology also has the property. Therefore the natural problem is to find the coarsest topology on $\mathcal{F}(X,Y)$ such that the evaluation map is continuous. Unfortunately, such coarsest topology may not always exist.

More generally, we may consider a collection $\mathcal{F}$ of maps from $X$ to $Y$. For example, in many applications, we need to consider the collection $\mathcal{F} = \mathcal{C}(X,Y)$ of all continuous maps from $X$ to $Y$. Moreover, for any subset $A \subset X$, we may consider (the restriction of) the evaluation map

$$E_A \colon \mathcal{F} \times A \to Y.$$

We say a topology on $\mathcal{F}$ is *jointly continuous* on $A$ if this restricted evaluation map is continuous. When we choose $A$ to be any compact subset of $X$, such continuity is related to the following topology.

**Definition 9.7.1.** The *compact-open topology* on the set $\mathcal{F}(X,Y)$ of maps from $X$ to $Y$ is induced by the topological subbasis consisting of

$$S(K,U) = \{f \in \mathcal{F}(X,Y) \colon f(K) \subset U\}, \quad K \subset X \text{ compact}, \quad U \subset Y \text{ open}.$$

It is easy to see that the product topology on $\mathcal{F}(X,Y)$ is induced by $S(\{x\},U)$ for $x \in X$ and open $U \subset Y$. Since single point spaces are compact, the product topology is coarser than the compact-open topology.

Exercise 9.7.1. Prove that if a collection $\mathcal{F}$ of maps is closed in the product topology, and $\mathcal{F}(x) = \{f(x) \colon f \in \mathcal{F}\}$ has compact closure in $Y$ for each $x \in X$, then $\mathcal{F}$ is compact in the product topology. Prove the converse is true if $Y$ is Hausdorff.

Exercise 9.7.2. Prove that if a collection $\mathcal{F}$ of maps has a topology finer than the product topology (compact-open topology, for example), then for any $x \in X$, the evaluation map $\mathcal{F} \to Y$, $f \mapsto f(x)$ is continuous.

Exercise 9.7.3. In case $X$ is a single point, $\mathcal{F}(X,Y)$ can be naturally identified with $Y$. Compare the compact-open topology on $\mathcal{F}(X,Y)$ with the topology of $Y$. Extend the discussion to the case $X$ is finite.

Exercise 9.7.4. Composing with a map $g \colon X \to Y$ induces a map

$$\circ g \colon \mathcal{F}(Y,Z) \to \mathcal{F}(X,Z),\ f \mapsto f \circ g.$$

1. Prove that $(\circ g)^{-1}(S(K,U)) = S(g(K),U)$.
2. Prove that $\circ g$ is continuous with respect to the product topology.
3. Prove that if $g$ is continuous, then $\circ g$ is continuous with respect to the compact-open topology.

Exercise 9.7.5. Composing with a map $g \colon Y \to Z$ induces a map

$$g\circ \colon \mathcal{F}(X,Y) \to \mathcal{F}(X,Z),\ f \mapsto g \circ f.$$

Prove that $(g\circ)^{-1}(S(K,U)) = S(K, g^{-1}(U))$. What does this tell you about the continuity of $g\circ$ with respect to the product or the compact-open topology?

Exercise 9.7.6. Prove that if $Y$ is Hausdorff, then the compact-open topology is also Hausdorff.

Exercise 9.7.7. Prove that if $\bar{V} \subset U$, then $\overline{S(K,V)} \subset S(K,U)$ in the compact-open topology. Use this to further prove that if $Y$ is regular, then the compact-open topology is also regular.

Exercise 9.7.8. Use Exercise 9.1.7 to show that, if $X$ is the discrete space of two points, then the compact-open topology on $\mathcal{F}(X, \mathbb{R}_{[\,)})$ is not normal, despite the fact that $\mathbb{R}_{[\,)}$ is a normal space.

**Proposition 9.7.2.** *If a collection $\mathcal{F}$ of maps from $X$ to $Y$ has a topology that is jointly continuous on any compact subset, then the topology is finer than the compact-open topology. Conversely, if $X$ is regular or Hausdorff and $\mathcal{F} \subset \mathcal{C}(X,Y)$ is a collection of continuous maps, then the compact-open topology on $\mathcal{F}$ is jointly continuous on any compact subset.*

*Proof.* For the necessity, we assume the topology on $\mathcal{F}$ is jointly continuous on any compact subset. We need to prove that $S(K,U)\cap\mathcal{F}$ is open in the given topology on $\mathcal{F}$.

For any $f\in S(K,U)\cap\mathcal{F}$ and $x\in K$, we have $E_K(f,x)=f(x)\in U$. Since $K$ is compact, the evaluation map $E_K\colon\mathcal{F}\times K\to Y$ is assumed to be continuous. Therefore there is an open neighborhood $B_x$ of $f$ in $\mathcal{F}$ and another open neighborhood $V_x$ of $x$ in $K$, such that

$$g\in B_x,\ v\in V_x \implies E_K(g,v)=g(v)\in U.$$

From the open cover $\{V_x\colon x\in K\}$ of the compact subset $K$, we have $K=V_{x_1}\cup V_{x_2}\cup\cdots\cup V_{x_k}$. Then for $g\in B=B_{x_1}\cap B_{x_2}\cap\cdots\cap B_{x_k}$ and $v\in K$, we have $g(v)\in U$. In other words, we have $B\subset S(K,U)$. Since $B$ is an open neighborhood of $f$ in the given topology of $\mathcal{F}$, this proves that $S(K,U)\cap\mathcal{F}$ is open in the given topology.

For the sufficiency, we first prove a general statement: If $\mathcal{F}$ consists of continuous maps and $X$ is regular and compact, then the compact-open topology on $\mathcal{F}$ is jointly continuous on $X$.

To prove the continuity of the evaluation $E\colon\mathcal{F}\times X\to Y$, we start with $E(f,x)=f(x)\in U$ for some $f\in\mathcal{F}$, $x\in X$, and open subset $U\subset Y$. By the continuity of $f$, we have $f(V)\subset U$ for some open neighborhood $V$ of $x$. Since $X$ is regular, there is an open neighborhood $W$ of $x$ satisfying $\bar{W}\subset V$. Since $X$ is compact, $\bar{W}$ is also compact. By $f(\bar{W})\subset f(V)\subset U$, therefore, $S(\bar{W},U)$ is a neighborhood of $f$ in the compact-open topology, and $(S(\bar{W},U)\cap\mathcal{F})\times W$ is an open neighborhood of $(f,x)$ in $\mathcal{F}\times X$ that is mapped into $U$ under the evaluation map. This proves the continuity of the evaluation map at $(f,x)$.

Now back to the sufficiency part of the proposition. Let $\mathcal{F}\subset\mathcal{C}(X,Y)$ and let $K\subset X$ be compact. Then by restricting maps in $\mathcal{F}$ to $K$, we get a collection $\mathcal{F}_K\subset\mathcal{C}(K,Y)$ (by taking $g$ in Exercise 9.7.4 to be the inclusion $K\to X$). If $K$ is regular, then by the general statement above, we conclude that, with the compact-open topology on $\mathcal{F}_K$, the evaluation map

$$\mathcal{F}_K\times K\to Y$$

is continuous. Moreover, by the third part of Exercise 9.7.4, the restriction map

$$\mathcal{F}\to\mathcal{F}_K,\ f\mapsto f|_K$$

is also a continuous map. Therefore we conclude that the evaluation map

$$E_K\colon\mathcal{F}\times K\to Y$$

is continuous.

It remains to verify the condition that the subspace topology on $K$ is regular. If $X$ is Hausdorff, then by Exercises 7.1.9 and 7.5.21, any compact subset of $X$ is regular. If $X$ is regular, then by Exercise 7.1.17, any subset is also regular. $\square$

The proposition shows that, for continuous maps (or at least continuous on any compact subset), the compact-open topology is directly related to the joint continuity on any compact subset. Exercises 9.7.9, 9.7.10 and 9.7.11 will tell us that, for locally compact or first countable spaces, the continuity (of a map) on the whole space is the same as the continuity on any compact subset. Exercise 9.7.13 will tell us that, for locally compact spaces, the joint continuity (of a collection of maps) on the whole space is equivalent to the joint continuity on any compact subset. Combined with Proposition 9.7.2, we find a sufficient condition for the compact-open topology to be the coarsest topology for the joint continuity on the whole space.

**Proposition 9.7.3.** *Suppose $X$ is regular or Hausdorff, and any point in $X$ has a compact neighborhood. Suppose $\mathcal{F} \subset \mathcal{C}(X,Y)$. Then the compact-open topology is the coarsest topology that is jointly continuous on the whole $X$.*

Exercise 9.7.9. Suppose $X$ is *compactly generated*: A subset $U$ is open if and only if $U \cap K$ is open in $K$ for any compact subset $K \subset X$.

1. Prove that a subset $C$ is closed if and only if $C \cap K$ is closed in $K$ for any compact subset $K \subset X$.
2. Prove that $f\colon X \to Y$ is continuous if and only if its restriction on any compact subset is continuous.

Exercise 9.7.10. Prove that if any point has a compact neighborhood (the weakest version of local compactness in Exercise 7.5.29), then the topology is compactly generated.

Exercise 9.7.11. Let $\{L_n\colon n \in \mathbb{N}\}$ be a countable local topological basis of $x$ satisfying $L_{n+1} \subset L_n$ (see Exercise 4.3.12). Let $x_n \in L_n$.

1. Prove that $\{x\} \cup \{x_n\colon n \in \mathbb{N}\}$ is compact.
2. Prove that $x \in \overline{\{x_n\colon n \in \mathbb{N}\}}$.
3. Prove that first countable spaces are compactly generated.

In particular, metric spaces are compactly generated.

Exercise 9.7.12. Prove that the joint continuity is local: A topology on $\mathcal{F}$ is jointly continuous on $A \subset X$ if and only if for any $a \in A$, there is an open subset $U \subset X$, such that $\mathcal{F}$ is jointly continuous on $A \cap U$.

Exercise 9.7.13. Suppose $X$ has the property that any point has a compact neighborhood (the weakest version of local compactness in Exercise 7.5.29). Prove that a topology on a collection $\mathcal{F}$ is jointly continuous on any compact subset of $X$ if and only if the topology is jointly continuous on the whole $X$.

The next result describes compact subsets in the compact-open topology.

**Proposition 9.7.4.** *If a collection $\mathcal{F}$ of maps from $X$ to $Y$ satisfies the following,*

1. *$\mathcal{F}$ is closed in the compact-open topology.*
2. *For any $x \in X$, the subset $\mathcal{F}(x) = \{f(x)\colon x \in X\}$ has compact closure in $Y$.*

3. *The product topology on the closure $\bar{\mathcal{F}}$ in the product topology is jointly continuous on any compact subset.*

*then $\mathcal{F}$ is compact in the compact-open topology. The converse is true if $X$ is regular or Hausdorff, $Y$ is Hausdorff, and $\mathcal{F} \subset \mathcal{C}(X, Y)$ consists of continuous maps.*

The proof below actually shows that the product topology on $\mathcal{F}$ is the same as the compact-open topology. Moreover, in case $Y$ is Hausdorff, the subset $\mathcal{F}(x)$ is already compact and closed, and the collection $\mathcal{F}$ is already closed in the product topology.

*Proof.* For the sufficiency, we need to prove that the three properties imply that the compact-open topology on $\mathcal{F}$ is compact.

We note that the collection $\mathcal{F}$ is contained in the product $\times_{x\in X}\mathcal{F}(x)$. By Exercise 9.5.4, the closure $\bar{\mathcal{F}}$ in the product topology is contained in $\times_{x\in X}\overline{\mathcal{F}(x)}$. By Theorem 9.5.2, the second condition implies that $\times_{x\in X}\overline{\mathcal{F}(x)}$ is compact in the product topology, so that the closed subset $\bar{\mathcal{F}}$ is also compact.

On the other hand, by Proposition 9.7.2, the third condition implies that the product topology on $\bar{\mathcal{F}}$ is finer than the compact-open topology. Since we also know that the compact-open topology is always finer than the product topology, the two topologies are actually the same on $\bar{\mathcal{F}}$. In particular, we know that the closure $\bar{\mathcal{F}}$ is also compact in the compact-open topology. Then the first condition tells us that $\mathcal{F}$ is a closed subset of $\bar{\mathcal{F}}$ in the compact-open topology. By Theorem 7.5.2, the compactness of $\bar{\mathcal{F}}$ in the compact-open topology implies the compactness of $\mathcal{F}$ in the compact-open topology.

For the necessity, we will prove a more general statement: Suppose $Y$ is Hausdorff. Suppose the compact-open topology on $\mathcal{F}$ is compact and is jointly continuous on any compact subset. Then $\mathcal{F}$ has the three properties.

Since $Y$ is Hausdorff, by Exercise 9.7.6, the compact-open topology on $\mathcal{F}(X, Y)$ is also Hausdorff. By Theorem 7.5.2, the compact subset $\mathcal{F}$ is closed. This verifies the first property.

For any $x \in X$, with the compact-open topology on $\mathcal{F}$, Exercise 9.7.2 tells us that the evaluation map $\mathcal{F} \to Y$, $x \mapsto f(x)$ is continuous. Since $\mathcal{F}$ is compact, by Theorem 7.5.2, the image $\mathcal{F}(x)$ of the evaluation map is compact. Since $\mathcal{F}(x) \subset Y$ and $Y$ is Hausdorff, by Theorem 7.5.2, $\mathcal{F}(x)$ is also closed. This verifies the second property.

Finally, since the product topology is coarser than the compact-open topology, the identity map

$$(\mathcal{F},\ \text{compact-open topology}) \to (\mathcal{F},\ \text{product topology})$$

is continuous. The left side is assumed to be compact. Since $Y$ is Hausdorff, by Exercise 9.5.5, the right side is also Hausdorff. Then by Theorem 7.5.2, the map is actually a homeomorphism. In other words, the product topology on $\mathcal{F}$ is the same as the compact-open topology on $\mathcal{F}$, which is assumed to be compact. Since the product topology on $\mathcal{F}(X, Y)$ is Hausdorff, by Theorem 7.5.2, the compact subset

$\mathcal{F}$ is also closed in the product topology. Therefore we have

$$(\bar{\mathcal{F}},\ \text{product topology}) = (\mathcal{F},\ \text{product topology}) = (\mathcal{F},\ \text{compact-open topology}).$$

Since the right side is assumed to be jointly continuous on any compact subset, the third property is verified.

Note that the joint continuity assumption is only used in the last step of the argument. Therefore if $\mathcal{F}$ is assumed to be jointly continuous on the whole $X$ in the general statement, then we may conclude that $(\bar{\mathcal{F}},$ product topology$)$ is jointly continuous on the whole $X$ in the third property.

As for the necessity as stated in the proposition, we simply note that, by Proposition 9.7.2, the assumptions in the proposition imply that the compact-open topology on $\mathcal{F}$ is jointly continuous on any compact subset. □

In case $Y$ is a metric space, we have a more explicit criterion for the compactness in the compact-open topology.

**Theorem 9.7.5** (Arzelà-Ascoli Theorem). *If a collection $\mathcal{F}$ of maps from $X$ to a metric space $Y$ satisfies the following,*

1. *$\mathcal{F}$ is closed in the compact-open topology.*
2. *For any $x \in X$, the subset $\mathcal{F}(x) = \{f(x) : x \in X\}$ has compact closure in $Y$.*
3. *$\mathcal{F}$ is equicontinuous: For any $x \in X$ and $\epsilon > 0$, there is an open subset $V$ around $x$, such that*

$$f \in \mathcal{F},\ v \in V \implies d(f(v), f(x)) < \epsilon.$$

*then $\mathcal{F}$ is compact in the compact-open topology. The converse is true if $X$ is locally compact and regular, and $\mathcal{F} \subset \mathcal{C}(X,Y)$ consists of continuous maps.*

For the discussion on locally compact and regular spaces, see Exercises 7.5.29 and 7.5.31. We remark that, by Exercise 7.5.33, locally compact and Hausdorff spaces are regular. Therefore the regular condition may be replaced by the Hausdorff condition.

Since $Y$ is Hausdorff, by the remark made after Proposition 9.7.4, the subset $\mathcal{F}(x)$ is actually compact and closed, the collection $\mathcal{F}$ is already closed in the product topology, and the product topology on $\mathcal{F}$ is the same as the compact-open topology.

The following is a more classical version of the Arzelá-Ascoli theorem. This version is often more useful for practical applications. Recall that a space is separable if it has a countable dense subset (see Example 5.2.9).

**Theorem 9.7.6** (Arzelà-Ascoli Theorem). *Suppose $X$ is a separable space and $Y$ is a metric space. If a collection $\mathcal{F}$ of maps from $X$ to $Y$ is equicontinuous, and the subset $\mathcal{F}(x) = \{f(x) : x \in X\}$ has compact closure in $Y$ for any $x \in X$, then any sequence in $\mathcal{F}$ has converging subsequence in the compact-open topology.*

In Exercises 9.7.14 through 9.7.18, Theorem 9.7.6 is derived from Theorem 9.7.5. In Exercises 9.7.20 and 9.7.21, Theorem 9.7.6 is proved directly.

*Proof of Theorem 9.7.5.* For the sufficiency, we need to verify the third condition in Proposition 9.7.4. This is a consequence of the following two facts.

1. If $\mathcal{F}$ is equicontinuous, then the closure $\bar{\mathcal{F}}$ in the product topology is equicontinuous.
2. If $\mathcal{F}$ is equicontinuous, then the product topology is jointly continuous on the whole $X$.

These facts actually hold without assuming the other two conditions.

Suppose for any $x \in X$ and $\epsilon > 0$, there is an open neighborhood $V$ of $x$, such that $f \in \mathcal{F}$ and $v \in V$ imply $d(f(v), f(x)) < \epsilon$. For any $g \in \bar{\mathcal{F}}$ and $v \in V$, the subsets $S(\{x\}, B(g(x), \epsilon))$ and $S(\{v\}, B(g(v), \epsilon))$ are open neighborhoods of $g$ in the product topology. By $g \in \bar{\mathcal{F}}$, there is $f$ lying in $\mathcal{F}$ and these two open neighborhoods. Then

$$d(g(v), g(x)) \le d(g(v), f(v)) + d(f(v), f(x)) + d(f(x), g(x)) < 3\epsilon.$$

This proves the first fact.

For the second fact, we fix $f \in \mathcal{F}$ and $x \in X$. For any $\epsilon > 0$, there is an open neighborhood $V$ of $x$, such that $g \in \mathcal{F}$ and $v \in V$ imply $d(g(v), g(x)) < \epsilon$. Then for $g \in \mathcal{F} \cap S(\{x\}, B(f(x), \epsilon))$ and $v \in V$, we have

$$d(g(v), f(x)) \le d(g(v), g(x)) + d(g(x), f(x)) < 2\epsilon.$$

In other words, the evaluation map $\mathcal{F} \times X \to Y$ takes the open neighborhood $(\mathcal{F} \cap S(\{x\}, B(f(x), \epsilon))) \times V$ of $(f, x)$ into $B(f(x), 2\epsilon)$. This proves that the product topology on $\mathcal{F}$ is jointly continuous.

For the necessity, we will prove a more general statement: If some topology on $\mathcal{F}$ is compact and jointly continuous on the whole $X$, then $\mathcal{F}$ is equicontinuous. We emphasis that the topology on $\mathcal{F}$ does not have to be the compact-open topology.

Fix $x \in X$ and $\epsilon > 0$. For any $f \in \mathcal{F}$, by the continuity of the evaluation map $E \colon \mathcal{F} \times X \to Y$ at $(f, x)$, there is an open neighborhood $B_f$ of $f$ in $\mathcal{F}$ and an open neighborhood $V_f$ of $x$, such that

$$g \in B_f,\ v \in V_f \implies g(v) \in B(f(x), \epsilon).$$

For the open cover $\{B_f \colon f \in \mathcal{F}\}$ of the compact subset $\mathcal{F}$, we have $\mathcal{F} = B_{f_1} \cup B_{f_2} \cup \cdots \cup B_{f_k}$. Then for $g \in \mathcal{F}$ and $v \in V = V_{f_1} \cap V_{f_2} \cap \cdots \cap V_{f_k}$, we have $g \in B_{f_i}$ and $v \in V_{f_i}$ for some $i$, so that $d(g(v), f_i(x)) < \epsilon$. Since $x$ also lies $V_{f_i}$, we also have $d(g(x), f_i(x)) < \epsilon$, and

$$d(g(v), g(x)) \le d(g(v), f_i(x)) + d(g(x), f_i(x)) < 2\epsilon.$$

This proves that $\mathcal{F}$ is equicontinuous.

As for the necessity as stated in the theorem, we simply note that, by Proposition 9.7.2 and Exercise 9.7.13, the assumptions in the theorem imply that the compact-open topology on $\mathcal{F}$ is compact and jointly continuous on the whole $X$. $\square$

The compact-open topology has a more explicit description in case $Y$ is a metric space and maps in $\mathcal{F}$ are continuous. In particular, this gives better understanding of Arzelà-Ascoli theorem. The proof is given as Exercises 9.7.23, 9.7.24, 9.7.25.

**Proposition 9.7.7.** *Suppose $Y$ is a metric space. Then the compact-open topology is the same as the topology induced by the topological basis*

$$B(f,K,\epsilon)=\{g\in\mathcal{C}(X,Y)\colon d(f(x),g(x))<\epsilon \text{ for all } x\in K\}$$

*where $K\subset X$ is compact and $\epsilon>0$.*

If $X$ is compact, then the compact-open topology is really induced by the $L_\infty$*-metric* on $\mathcal{C}(X,Y)$

$$d_\infty(f,g)=\max_{x\in X} d(f(x),g(x)).$$

This extends the $L_\infty$-metric on $C[0,1]$. The convergence (of a sequence, for example) with respect to such a metric is the *uniform convergence*. In general, therefore, the convergence with respect to the compact-open topology is the uniform convergence over any compact subset.

Exercise 9.7.14. Prove that the closure $\bar{\mathcal{F}}$ in the product topology satisfies $\bar{\mathcal{F}}(x)\subset\overline{\mathcal{F}(x)}$.

Exercise 9.7.15. Prove that on any equicontinuous collection, the product topology and the compact-open topology are the same.

Exercise 9.7.16. Prove that if $\mathcal{F}$ is equicontinuous and $A$ is a dense subset of $X$, then the product topology on $\mathcal{F}$ (as well as the compact-open topology, by Exercise 9.7.15) is induced by the topological basis $S(\{a\},U)\cap\mathcal{F}$ for $a\in A$ and open $U$. In other words, the pointwise convergence for maps in $\mathcal{F}$ is the same as the convergence at each point of $A$.

Exercise 9.7.17. Prove that if $\mathcal{F}$ is equicontinuous and $X$ is separable, then the product topology on $\mathcal{F}$ (as well as the compact-open topology) is first countable.

Exercise 9.7.18. With the help of Theorem 7.6.2 and Exercises 7.6.16, 9.7.14, 9.7.15, 9.7.17, derive Theorem 9.7.6 from Theorem 9.7.5.

Exercise 9.7.19. Suppose $X$ is a separable space and $\mathcal{F}$ is a collection of functions on $X$ with values in the Euclidean space $\mathbb{R}^n$. Suppose $\mathcal{F}$ is equicontinuous and pointwise bounded: For any $x\in X$, $\mathcal{F}(x)$ is a bounded subset of $\mathbb{R}^n$. Prove that any sequence in $\mathcal{F}$ has a subsequence that uniformly converges on any compact subset of $X$.

Exercise 9.7.20. Suppose $f_n$ is a sequence of maps, such that the value $\{f_n(x)\colon n\in\mathbb{N}\}$ of the sequence at each $x$ has sequentially compact closure. Suppose $A\subset X$ is a countable subset. Use Exercise 9.5.14 to prove that $f_n$ has a subsequence converging at each point of $A$.

Exercise 9.7.21. Prove that if a sequence of maps $f_n$ is equicontinuous and converges at each point of a dense subset, then the sequence also converges in the compact-open topology.

Exercise 9.7.22. Suppose $f_n$ is an equicontinuous sequence of maps pointwise converging (i.e. converging in the product topology) to $f$. Prove that $f$ is also continuous, and the sequence also converges in the compact-open topology.

Exercise 9.7.23. Prove that for $g \in B(f, K, \epsilon)$, there is $\delta > 0$, such that $B(g, K, \delta) \subset B(f, K, \epsilon)$. Then use this to prove that the subsets $B(f, K, \epsilon)$ form a topological basis.

Exercise 9.7.24. Suppose $Y$ is a metric space, $f\colon X \to Y$ is continuous, $K \subset X$ is compact, and $U \subset Y$ is open. Use Exercise 7.5.20 to find $\epsilon > 0$, such that the $\epsilon$-neighborhood of $f(K)$ is contained in $U$. Explain why your result means $B(f, K, \epsilon) \subset S(K, U)$ and, with the help of Exercise 9.7.23, prove that the topology induced by the topological basis $\{B(f, K, \epsilon)\}$ is finer than the compact-open topology.

Exercise 9.7.25. Suppose $Y$ is a metric space and $f\colon X \to Y$ is continuous on a compact subset $K$. Prove that for any $\epsilon > 0$, there are finitely many $x_1, x_2, \dots, x_k \in K$ and open neighborhoods $V_1, V_2, \dots, V_k$ of these points in $K$, such that $K = V_1 \cup V_2 \cup \dots \cup V_k$ and (the closures are taken in $K$)

$$S(\bar{V}_1, B(f(x_1), \epsilon)) \cap S(\bar{V}_2, B(f(x_2), \epsilon)) \cap \dots \cap S(\bar{V}_k, B(f(x_k), \epsilon)) \subset B(f, K, \epsilon).$$

Then with the help of Exercise 5.1.23, prove that the compact-open topology is finer than the topology induced by the topological basis $\{B(f, K, \epsilon)\}$.

Exercise 9.7.26. Suppose $X$ is $\sigma$*-compact*: $X$ is covered by countably many open subsets, each contained in a compact subset $K_n$. Suppose $Y$ is a bounded metric space. Prove that

$$d(f, g) = \sum_{n=1}^{\infty} \frac{\max_{x \in K_n} d(f(x), g(x))}{2^n}$$

is a metric on $\mathcal{C}(X, Y)$ inducing the compact-open topology. Since the topology of a metric space is always induced by a bounded metric (see Exercise 4.4.1), the compact-open topology is induced by a metric even if the metric on $Y$ is not assumed to be bounded.

Exercise 9.7.27. Suppose $X$ is $\sigma$-compact and $Y$ is a complete metric space. Prove that the compact-open topology on $\mathcal{C}(X, Y)$ is induced by a complete metric.

Exercise 9.7.28. Suppose $X$ is $\sigma$-compact and $Y$ is a metric space. Suppose a sequence of maps $f_n\colon X \to Y$ is equicontinuous, and the value $\{f_n(x)\colon n \in \mathbb{N}\}$ of the sequence at each $x$ has compact closure. Prove that $f_n$ has a subsequence that uniformly converges on any compact subset.

Next we study maps involving more than two spaces. We always have a one-to-one correspondence

$$\mathcal{F}(X \times Y, Z) \to \mathcal{F}(X, \mathcal{F}(Y, Z)),\ f \mapsto \varphi,\ \text{where } \varphi(x)(y) = f(x, y).$$

Essentially, this identifies a map $f$ of two variables as a family $\varphi$ of maps in one variable, and the family itself is also considered as a map.

**Proposition 9.7.8.** *If $f$ is continuous, then $\varphi(x)\colon Y \to Z$ is continuous for each $x$, and $\varphi\colon X \to \mathcal{C}(Y,Z)$ is also continuous, where $\mathcal{C}(Y,Z)$ has the compact-open topology. Conversely, if $Y$ has the property that any neighborhood of a point contains a compact neighborhood of the point, then the continuity of $\varphi(x)\colon Y \to Z$ for each $x$ and the continuity of $\varphi\colon X \to \mathcal{C}(Y,Z)$ imply the continuity of $f$.*

The technical condition that any neighborhood of a point contains a compact neighborhood of the point appeared in Exercise 7.5.29, and is one of several versions of local compactness. For Hausdorff or regular spaces, the different versions are equivalent (see also Exercise 7.5.31).

*Proof.* Suppose $f$ is continuous. The map $\varphi(x)$ is continuous for each $x$ because the inclusion map $Y \to x \times Y \subset X \times Y$ is continuous. To prove the continuity of $\varphi$, we only need to prove that $\varphi^{-1}(S(K,U))$ is open for any compact $K \subset Y$ and open $U \subset Z$. In other words, if $f(x \times K) \subset U$, then there is an open neighborhood $V$ of $x$, such that $f(V \times K) \subset U$. Since $f$ is continuous, we see that $W = f^{-1}(U)$ is an open subset of $X \times Y$ containing $x \times K$. We simply need to find open $V$, such that $x \in V$ and $V \times K \subset W$. This is essentially proved in the sixth property in Theorem 7.5.2. Specifically, for any $y \in K$, we have $(x,y) \in W$. Since $W$ is open, we have $(x,y) \subset V_y \times V'_y \subset W$ for some open subsets $V_y \subset X$ and $V'_y \subset Y$. By $K \subset \cup_y V'_y$ and $K$ compact, we have $K \subset V'_{y_1} \cup V'_{y_2} \cup \cdots \cup V'_{y_n}$. Then for the open neighborhood $V = V_{y_1} \cap V_{y_2} \cap \cdots \cap V_{y_n}$ of $x$, we have $V \times K \subset W$.

Conversely, assume $Y$ has the property that any neighborhood of a point contains a compact neighborhood of the point. Assume the map $\varphi(x)$ is continuous for each $x$, and the map $\varphi$ is continuous. To prove the continuity of $f$, consider $f(x,y) \in U$ for some open $U \subset Z$. By $\varphi(x)(y) = f(x,y) \in U$ and the continuity of $\varphi(x)$, $\varphi(x)$ maps a neighborhood of $y$ into $U$. By the special property of $Y$, we may assume that the neighborhood is a compact subset $K$ and get $\varphi(x) \in S(K,U)$. Then by the continuity of $\varphi$, there is an open neighborhood $V$ of $x$, such that $\varphi(V) \subset S(K,U)$. The inclusion simply means $f(V \times K) \subset U$. Since $V \times K$ is a neighborhood of $(x,y)$, this proves the continuity of $f$. □

Exercise 9.7.29. Suppose $X$ has the property that any neighborhood of a point contains a compact neighborhood of the point. By applying Proposition 9.7.8 to the case $X$, $Y$, $Z$, $f$ are replaced by $\mathcal{C}(X,Y)$, $X$, $Y$, $E$, prove that the evaluation map $E\colon \mathcal{C}(X,Y) \times X \to Y$ is continuous.

Exercise 9.7.30. Suppose $X$ and $Y$ have the property that any neighborhood of a point contains a compact neighborhood of the point. Prove that the composition map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$$

is continuous.

Exercise 9.7.31. Proposition 9.7.8 can be used to prove the following: If $f\colon X \to Y$ is a quotient map, and $Z$ is locally compact and regular, then $f \times id\colon X \times Z \to Y \times Z$ is also a quotient map. Denote the quotient topologies by $Y_q$ and $(Y \times Z)_q$. Prove the claim in the following steps.

1. Prove that $(Y \times Z)_q \to Y_q \times Z$ is continuous.
2. Prove that the natural map $X \to \mathcal{C}(Z, X \times Z)$ is continuous.
3. Prove that $Y_q \to \mathcal{C}(Z, (Y \times Z)_q)$ is continuous.
4. Prove that $Y_q \times Z \to (Y \times Z)_q$ is continuous.

Proposition 9.7.8 tells us that there is indeed a map in general

$$\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

Moreover, the map is onto (and therefore a one-to-one correspondence) if $Y$ is locally compact as described in the proposition. The next natural question is whether the map is continuous or homeomorphic, with compact-open topology used throughout.

**Proposition 9.7.9.** *The natural map $\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$ is a homeomorphism if both $X$ and $Y$ have the property that any neighborhood of a point contains a compact neighborhood of the point.*

*Proof.* By thinking of the natural map as lying in

$$\mathcal{F}(\mathcal{C}(X \times Y, Z), \mathcal{C}(X, \mathcal{C}(Y, Z))),$$

we may apply Proposition 9.7.8 to the case $X$, $Y$ and $Z$ are replaced by $\mathcal{C}(X \times Y, Z)$, $X$ and $\mathcal{C}(Y, Z)$. Thus the continuity of the natural map $\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$ is reduced to the continuity of the related map

$$\mathcal{C}(X \times Y, Z) \times X \to \mathcal{C}(Y, Z), \ (f, x) \mapsto \varphi(x).$$

We may yet again apply Proposition 9.7.8 to the case $X$, $Y$ and $Z$ are replaced by $\mathcal{C}(X \times Y, Z) \times X$, $Y$ and $Z$. This further reduces the problem to the continuity of the evaluation map

$$\mathcal{C}(X \times Y, Z) \times X \times Y \to Z.$$

By Theorem 7.5.2, the product $X \times Y$ still has the property that any neighborhood of a point contains a compact neighborhood of the point. Therefore we can further apply the converse part of Proposition 9.7.8 and reduce the problem to the continuity of the related map

$$\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X \times Y, Z).$$

This is the identity map and is therefore continuous.

Similarly, the inverse of the natural map can be considered as lying in

$$\mathcal{F}(\mathcal{C}(X, \mathcal{C}(Y, Z)), \mathcal{C}(X \times Y, Z)).$$

We may apply Proposition 9.7.8 to reduce the problem to the continuity of the related map

$$\mathcal{C}(X, \mathcal{C}(Y, Z)) \times X \times Y \to Z, \ (\varphi, x, y) \mapsto \varphi(x)(y).$$

The map is a composition

$$\mathcal{C}(X, \mathcal{C}(Y, Z)) \times X \times Y \to \mathcal{C}(Y, Z) \times Y \to Z$$

of evaluation maps, which are continuous by Exercise 9.7.29. □

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